TRANSMISSION NETWORKS AND CIRCUITS

Theory, worked examples and problems

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PREFACE

The author has for some time found great difficulty in recommending to students studying the subject of 'circuit theory', particularly degree level, a textbook of problems for supplementing their lecture notes during private study.

The problems here have been chosen to avoid repetition as far as possible; some questions require standard proofs while others have been included to give a general picture of the type of questions asked in examinations.

To assist students in their reading, a brief theoretical introduction is given at the start of each chapter together with a general reference to textbooks found most suitable to the topics involved in the book. The list is by no means exhaustive but represents books which the author has found readable and suitable to a study of circuit theory.

The author wishes to express her gratitude to the Council of Engineering Institutions, for permission to use questions from their examination papers - the answers given here are the entire responsibility of the author.

R. V. BUCKLEY

CONTENTS

1 TRANSMISSION LINES

The series resistance and inductance and the leakage conductance **and capacitance to earth** of a transmission line are distributed **over the entire length of the line.** The analysis of circuits of **this type cannot be carried out** in the same way as circuits in **which the parameters are lumped** together. The relationship **between the currents** and voltages at any point on a transmission **line and the currents** and voltages which appear at the load may **be seen as follows.** Consider a generator feeding a load Z_L at a **frequency** ω **radians** per second through a line of over-all length **1, which has distributed** series impedance $(R + j\omega L)\Omega$ per unit **length and a distributed** shunt admittance of $(G + j\omega C)$ siemens per **unit length.** (See figure 1.1.)

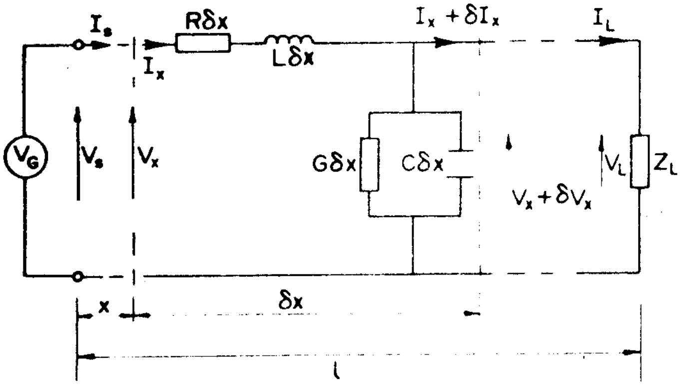

Figure 1.1

Apply Kirchhoff's law to the section

$$V_x - (V_x + \delta V_x) = (R + j\omega L)\delta x\ I_x$$

assuming constant current

$$(R + j\omega L) I_x = -\frac{\delta V_x}{\delta x} \qquad (1.1)$$

$$I_x - (I_x + \delta I_x) = (G + j\omega C)\delta x V_x$$

assuming constant voltage

$$(G + j\omega C)V_x = -\frac{\delta I_x}{\delta x} \qquad (1.2)$$

Partial derivatives are used to indicate that only the variations with distance are being considered. The time variations are declared in the use of the complex notation for impedance and admittance or alternatively by the use of a sinusoidal signal from the generator.

By differentiation and substitution

$$\frac{\delta^2 V_x}{\delta x^2} - ZYV_x = 0 \qquad (1.3)$$

where $Z = R + j\omega L$ and $Y = G + j\omega C$. Similarly

$$\frac{\delta^2 I_x}{\delta x^2} - ZYI_x = 0 \qquad (1.4)$$

The product $ZY = \gamma^2$, the propagation coefficient, and for a given line γ is constant but may be complex.

$$\gamma = \alpha + j\beta \qquad (1.5)$$

where α is the attenuation per unit length, in nepers, and β is the phase change per unit length, in radians.

In the limit as $\delta x \rightarrow 0$ equation 1.3 yields as a solution

$$V_x = Ae^{-\gamma x} + Be^{+\gamma x} \qquad (1.6)$$

Differentiation and substitution into equation 1.1 give

$$I_x = \frac{A}{Z_0} e^{-\gamma x} - \frac{B}{Z_0} e^{+\gamma x} \qquad (1.7)$$

where

$$Z_0 = \left(\frac{Z}{Y}\right)^{\frac{1}{2}} = \left(\frac{R + j\omega L}{G + j\omega C}\right)^{\frac{1}{2}} \qquad (1.8)$$

and is known as the characteristic impedance of the line. Equations 1.6 and 1.7 also show that two travelling waves are on the line known as the forward and reflection waves. Therefore to find the complex constants A and B, consider when $x = 0$ then $V_x = V_s$ and $I_x = I_s$. Therefore

$$A = \frac{V_s + I_s Z_0}{2} \quad \text{and} \quad B = \frac{V_s - I_s Z_0}{2}$$

therefore

$$I_x = \left(\frac{V_s + I_s Z_0}{2}\right) e^{-\gamma x} + \left(\frac{V_s - I_s Z_0}{2}\right) e^{+\gamma x} \qquad (1.9)$$

$$I_x = \left(\frac{V_s + I_s Z_0}{2Z_0}\right) e^{-\gamma x} - \left(\frac{V_s - I_s Z_0}{2}\right) e^{+\gamma x} \qquad (1.10)$$

In hyperbolic form and putting $x = \ell$

$$V_L = V_s \cosh \gamma\ell - I_s Z_0 \sinh \gamma\ell \qquad (1.11)$$

$$I_L = I_s \cosh \gamma\ell - \frac{V_s}{Z_0} \sinh \gamma\ell \qquad (1.12)$$

If considering the problem from the load end, x may be taken as negative, then

$$V_s = V_L \cosh \gamma\ell + I_L Z_0 \sinh \gamma\ell \qquad (1.13)$$

$$I_s = I_L \cosh \gamma\ell + \frac{V_L}{Z_0} \sinh \gamma\ell \qquad (1.14)$$

Thus the input impedance to the line is

$$Z_s = Z_{input} = \frac{V_s}{I_s} = Z_0 \left(\frac{Z_L \cosh \gamma\ell + Z_0 \sinh \gamma\ell}{Z_0 \cosh \gamma\ell + Z_L \sinh \gamma\ell}\right) \qquad (1.15)$$

From equation 1.15 it can be seen that if $Z_L = Z_0$ then

$$Z_{input} = Z_0$$

Similarly from equations 1.9 or 1.10, if $V_s = I_s Z_s = I_s Z_0$ for $Z_s = Z_0$, that is, the line is matched and the reflected travelling wave

$$\left(\frac{V_s - I_s Z_0}{2}\right) e^{+\gamma x}$$

disappears.

$$V_x = V_s e^{-\gamma x} \quad \text{and} \quad I_x = I_s e^{-\gamma x} \qquad (1.16)$$

$$V_x = V_s e^{-\alpha x} e^{-j\beta x} \quad \text{and} \quad I_x = I_s e^{-\alpha x} e^{-j\beta x}$$

This expression represents a rotating vector, showing the diminishing value of voltage $V_s e^{-\alpha x}$ and rotating through an angle βx - lagging

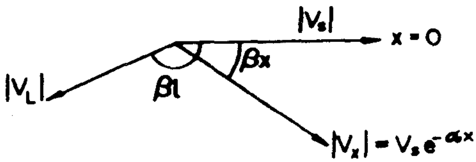

Figure 1.2

because of the negative sign. (See figure 1.2.)

The waves travel at the speed of light in loss-free lines only - the actual velocity of propagation being given by

$$\text{velocity} = \text{wavelength } \lambda \times \text{frequency } f$$

$$= \lambda \frac{\omega}{2\pi} \tag{1.17}$$

but $\lambda = (2\pi)/\beta$, therefore

$$\text{velocity} \doteq \frac{\omega}{\beta} \text{ m/s} \tag{1.18}$$

Example 1.1

A single-phase transmission line 10 miles long has the following parameters

resistance per loop mile 50 Ω

inductance per mile 0.001 H

capacitance per mile 0.06 μF

The shunt conductance may be neglected.

Calculate the characteristic impedance of the line. If the impedance of the load is equal to the characteristic impedance of the line and a potential difference of 5 volts at a frequency of 5000/(2π) Hz is applied at the sending end, calculate

(a) the magnitude of the received current
(b) the wavelength
(c) the velocity of propagation.

$$Z_0 = 10^3 \left(\frac{50 + j5000 \times 0.002}{j5000 \times 0.03}\right)^{\frac{1}{2}} = 10^3 \left(\frac{50 + j10}{j150}\right)^{\frac{1}{2}}$$

$$Z_0 = 583 \underline{/-39.3^\circ} \ \Omega$$

Since the line is matched

$$I_L = I_s e^{-\gamma\ell} \quad \text{and} \quad V_L = V_s e^{-\gamma\ell}$$

Now $\gamma = [(50 + j10)j150 \times 10^{-6}]^{\frac{1}{2}} = 0.0875 \underline{/50.75°}$

$$\gamma\ell = 0.554 + j0.678$$

therefore

$$V_L = 5e^{-0.554}\, e^{-j0.678} = 2.87 \underline{/-38.85°}$$

$$I_L = \frac{2.87 \underline{/-38.85°}}{583 \underline{/39.30°}} = 4.92 \text{ mA}$$

$$\text{Velocity of propagation} = \frac{5000}{0.0678} \text{ mile/s}$$

$$= 73\,746 \text{ mile/s}$$

$$\text{Wavelength } \lambda = \frac{2\pi}{0.0678} = 92.7 \text{ mile}$$

Example 1.2

The impedance of a telephone line is measured with the far end on open-circuit and found to be 750 $\underline{/57°}$ Ω, and with the far end short-circuited 480 $\underline{/-63°}$ Ω. Calculate the input voltage needed to supply 2 mW to a resistive load of 300 Ω.

From the input impedance equation 1.15 and the information that on open-circuit $Z_L = \infty$ and on short-circuit $Z_L = 0$, then

$$Z_{oc} = \frac{Z_0}{\tanh \gamma\ell} \quad \text{and} \quad Z_{sc} = Z_0 \tanh \gamma\ell$$

giving

$$Z_0 = \left(Z_{oc} \times Z_{sc}\right)^{\frac{1}{2}} \quad \text{and} \quad \tanh \gamma\ell = \left(\frac{Z_{sc}}{Z_{oc}}\right)^{\frac{1}{2}}$$

therefore

$$Z_0 = (750 \underline{/57°} \times 480 \underline{/-63°})^{\frac{1}{2}} = 600 \underline{/-3°}\ \Omega$$

$$\tanh \gamma l = \left(\frac{480 \angle{-63^\circ}}{750 \angle{57^\circ}}\right)^{\frac{1}{2}} = 0.8 \angle{-60^\circ}$$

Using the hyperbolic relationship

$$\cosh \gamma l = \frac{1}{(1 - \tanh^2 \gamma l)^{\frac{1}{2}}} = 0.89 \angle{-11.4^\circ}$$

$$\sinh \gamma l = \frac{\tanh \gamma l}{(1 - \tanh^2 \gamma l)^{\frac{1}{2}}} = 0.71 \angle{-71.4^\circ}$$

also $I_L{}^2 R$ = power, therefore

$$I_L = \left(\frac{2 \times 10^{-3}}{300}\right)^{\frac{1}{2}} = 2.58 \text{ mA}$$

and $V_L = 300 \times 2.58 \times 10^{-3} = 0.774$ volts

therefore

$$V_s = 0.774 \times 0.89 \angle{-11.4^\circ} + 2.58 \times 10^{-3} \times 600 \angle{-3^\circ} \times 0.71 \angle{-71.4^\circ}$$

$$V_s = 0.68 - j0.136 + 0.296 - j1.06$$

$$V_s = 0.976 - j1.196 = 1.54 \angle{-50.9^\circ} \text{ volts}$$

Example 1.3

A loss-free transmission line of characteristic impedance 50 Ω is terminated at one end in a short-circuit and at the other end in a resistive impedance of 85 Ω. (See figure 1.3.) The impedance

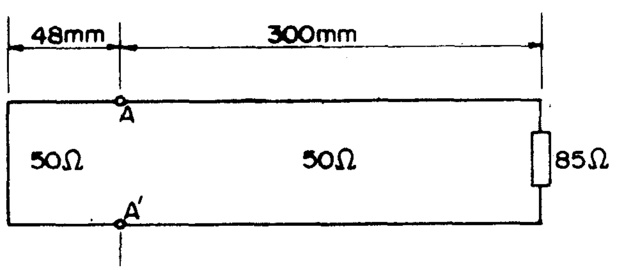

Figure 1.3

measured at the junction AA' is found to be 75 Ω, resistive, at a frequency of 44 MHz. Calculate the phase velocity in the transmission line.
[C.E.I. Part 2, E.F.C., 1968]

$$Z_{input} = Z_0 \left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}\right).$$

since the line is loss free and $\gamma = 0 + j\beta$, that is, $\alpha = 0$. On short-circuit $Z_L = 0$, $Z_{input} = jZ_0 \tan \beta l$

$$Z_{input} = Z_{sc} = j50 \tan \beta \times 0.048 \qquad (1)$$

Similarly for the 300 mm line

$$Z_{input} \quad 50 \left(\frac{85 + j50 \tan 0.3\beta}{50 + j85 \tan 0.3\beta}\right) \qquad (2)$$

The parallel combination of these two impedances is 75 Ω. However, it is easy to work in admittance

$$\frac{1}{75} = \frac{-j}{50 \tan \beta 0.048} + \frac{1}{50}\left(\frac{50 + j85 \tan 0.3\beta}{85 + j50 \tan 0.3\beta}\right) \qquad (3)$$

$$\frac{1}{75} = \frac{-j}{50 \tan \beta 0.048} + \left(\frac{1 + j1.7 \tan 0.3\beta}{85^2 + 50^2 \tan^3 0.3\beta}\right) (85 - j50 \tan 0.3\beta)$$

equate real terms only

$$\frac{1}{75} = 85 \frac{(1 + \tan^2 0.3\beta)}{85^2 + 50^2 \tan^2 0.3\beta} \qquad (4)$$

therefore

$$85^2 + 50^2 \tan^2 0.3\beta = 75 \times 85 + 75 \times 85 \tan^2 0.3\beta$$

$$3875 \tan^2 0.3\beta = 850$$

$$\tan^2 0.3\beta = 0.219 \quad \tan 0.3\beta = 0.4683$$

$$0.3\beta = 25.1°$$

$$\beta = 83.67 \times \frac{\pi}{180} = 1.46 \text{ radians} \qquad (5)$$

$$\text{Velocity of propagation} = \frac{\omega}{\beta} = \frac{2\pi \times 44 \times 10^6}{1.46} = 1.89 \times 10^8 \text{ m/s} \qquad (6)$$

Example 1.4

Derive an expression for the reflection coefficient ρ in terms of load and characteristic impedance, using the complexors V^+ and V^- representing the forward and reflected voltage parameters instead of the usual complex parameters. Hence show that the voltage V_x can be expressed in the form

$$V^+ (\underline{/0°} + |\rho| \underline{/\theta° - 4\pi x/\lambda°})$$

when operating from the load end.

$$V_x = V^+ e^{+\gamma x} + V^- e^{-\gamma x}$$

where

$$V^+ = \frac{V_L + I_L Z_0}{2} \quad \text{and} \quad V^- = \frac{V_L - I_L Z_0}{2}$$

and

$$I_x = \frac{V^+}{Z_0} e^{\gamma x} - \frac{V^-}{Z_0} e^{-\gamma x}$$

where γ is the propagation coefficient per unit length of line and Z_0 is the characteristic impedance.

At the load x = 0

$$V_L = V^+ + V^- \quad \text{and} \quad I_L = \frac{V^+}{Z_0} - \frac{V^-}{Z_0}$$

by adding these equations we obtain

$$2V^+ = V_L + I_L Z_0$$

and by subtracting

$$2V^- = V_L - I_L Z_0$$

By definition the reflection coefficient is the ratio of the reflection voltage wave to the forward voltage wave.

In the general case

$$\rho = \frac{V^- e^{-\gamma x}}{V^+ e^{\gamma x}} = \frac{V^-}{V^+} e^{-2\gamma x}$$

(Note the exponential index of $-2\gamma x$.) At the load where x = 0

$$\rho = \frac{V^-}{V^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Reminder: ρ will normally be complex, that is, $|\rho| \underline{/\theta}$, therefore

$$V_x = V^+ e^{+\gamma x} + \rho V^+ e^{-\gamma x}$$

$$I_x = \frac{V^+}{Z_0} e^{+\gamma x} - \rho \frac{V^+}{Z_0} e^{-\gamma x}$$

also $\frac{V^+}{I^+} = + Z_0$ and $\frac{V^-}{I^-} = - Z_0$

therefore

$$I^- = -\rho I^+$$

that is, current reflection coefficient equal and opposite to that of the voltage. Note also that $I_L = I^+ + I^-$.

For a loss-free line $e^{\pm\gamma x}$ becomes $e^{\pm j\beta x}$, therefore

$$V_x = V^+ \left(e^{+j\beta x} + |\rho| \underline{/\theta}\, e^{-j\beta x}\right)$$

$$V_x = V^+ \left(\underline{/\beta x} + |\rho| \underline{/\theta - \beta x}\right)$$

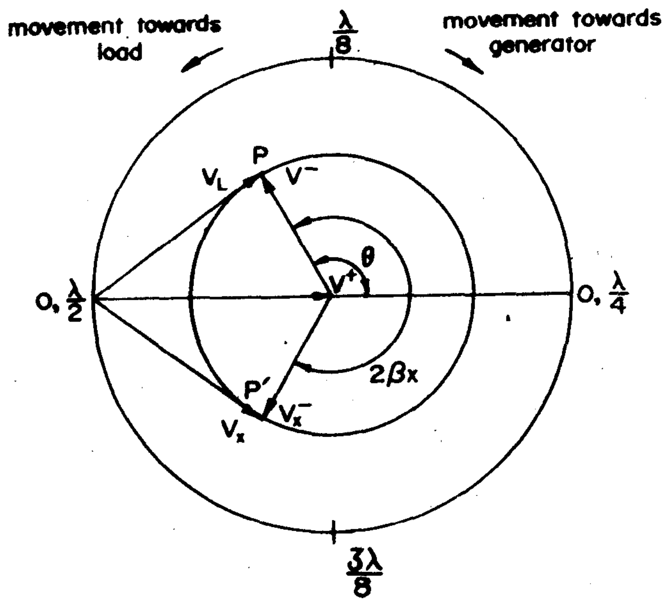

Figure 1.4

Draw a vector diagram; taking the forward wave V^+ as reference then, to maintain the correct phase position between V^+ and V^-, the reflected wave rotates through twice βx. (See figure 1.4.) Therefore

$$V_x = V^+ \left(\underline{/0^\circ} + |\rho| \underline{/\theta^\circ - 2\beta x^\circ}\right)$$

but $\beta = 2\pi/\lambda$, therefore

$$V_x = V^+ \left(\underline{/0^\circ} + |\rho| \; \underline{/\theta^\circ - 4\pi x/\lambda^\circ} \right)$$

Note that the first maximum voltage occurs when $\theta^\circ = 2\beta x^\circ$ while the first minimum voltage from the load occurs when $\theta^\circ + 180^\circ = 2\beta x^\circ$, for this particular diagram.

Example 1.5

From the information developed in example 1.4, explain what is meant by a standing wave and develop an expression for the V.S.W.R. in terms of ρ.

First of all note the circle drawn with V^+ as radius in figure 1.4, together with the circle drawn with V^- as radius. As x increases from point P (the load) towards the source, the phase angle of the reflected wave increases in the negative sense (clockwise) by $(2\pi/\lambda) \times 2x$, so that point P moves round the circle of radius V^- or $|\rho|V^+$. Each time x increases by $\lambda/2$, the phase angle changes by 2π radians, so that a linear scale can be marked on the outside circle representing wavelength moved - once round being equivalent to half a wavelength. The peak magnitude of the alternating voltage, V_x at any point x, on the line is seen to vary with x. The line is said to support a standing wave due to the interaction between the forward and reflected travelling waves.

$$V_{max} = V^+ + |\rho|V^+$$

$$V_{min} = V^+ - |\rho|V^+$$

The ratio of this maximum value to the minimum value is called the voltage standing wave ratio (V.S.W.R.)

$$S = \text{V.S.W.R.} = \frac{V_{max}}{V_{min}} = \frac{1 + |\rho|}{1 - |\rho|}$$

$$\text{or} \quad |\rho| = \frac{S - 1}{S + 1}$$

Clearly in the case of a matched line when $V^- = 0, S = 1$.

Example 1.6

A line of characteristic impedance $600\underline{/0^\circ}\ \Omega$ is terminated in a load Z_L. The V.S.W.R. measured on the line is 1.5 and the first maximum occurs at 20 cm from the load. The line is open wire and is supplied from a generator at 300 MHz. Find the value of the load impedance.

Since the line is open wire

$$3 \times 10^8 = \lambda \times 300 \times 10^6$$

or $\lambda = 1\text{m}$

therefore 20 cm is equivalent to 0.2λ, thus $x = 0.2\lambda$.

From example 1.5

$$|\rho| = \frac{1.5 - 1}{1.5 + 1} = 0.2$$

while the angle is obtained from

$$\theta^\circ = 2\beta x^\circ = \frac{4\pi}{\lambda} \times 0.2\lambda$$

$$\theta = 0.8\pi \text{ radians} = 144^\circ$$

This positive angle indicates that the reflected wave leads the forward wave at the load by 144°. Therefore

$$0.2 \underline{/144^\circ} = \frac{Z_L - 600}{Z_L + 600}$$

$$Z_L (1 + 0.162 - j0.117) = 600 (-0.162 + j0.117 + 1)$$

$$Z_L = 600 \left(\frac{0.838 + j0.117}{1.162 - j0.117}\right)$$

$$= \frac{600 \times 0.846 \underline{/8^\circ}}{1.168 \underline{/-5.75^\circ}}$$

$$= 434.6 \underline{/13.75^\circ}$$

$$Z_L = (422 + j103.3)\ \Omega$$

Example 1.7

Show that a loss-free transmission line of length [(λ/4) + (nλ/2)] may be referred to as an impedance transformer and is used as an impedance-matching device.

Such a line has a characteristic impedance of 600 Ω and negligible losses. It is 50 m long and is open-circuited at one end. At the other end it is connected to a generator which generates 100 volts at 15 MHz, and has an internal impedance of (200 + j200) Ω. Non-inductive loads of 600 Ω each are connected across the line at distances of 30 and 35 m from the source. Calculate the currents in the two loads.

The input impedance at any distance x down the line is given by

$$Z_{input} = Z_0 \left[\frac{Z_L \cosh \gamma x + Z_0 \sinh \gamma x}{Z_0 \cosh \gamma x + Z_L \sinh \gamma x}\right]$$

For a loss-free line, $\gamma x = j\beta x$, therefore

$$Z_{input} = Z_0 \left[\frac{Z_L \cos \beta x + jZ_0 \sin \beta x}{Z_0 \cos \beta x + jZ_L \sin \beta x}\right]$$

Divide through by $\cos \beta x$

$$Z_{input} = Z_0 \left[\frac{Z_L + jZ_0 \tan \beta x}{Z_0 + jZ_L \tan \beta x}\right]$$

Reminder: $Z_0 = (L/C)^{\frac{1}{2}}$, a real quantity, and $\beta = (LC)^{\frac{1}{2}}$, **therefore**

$$Z_{oc} = \frac{Z_0}{j \tan \beta x} = \frac{-jZ_0}{\tan \beta x} = -j\left(\frac{L}{C}\right)^{\frac{1}{2}} \cot \omega\sqrt{(LC)}\,x$$

$$Z_{oc} = -jZ_0 \cot \frac{2\pi x}{\lambda}$$

Similarly

$$Z_{sc} = jZ_0 \tan \frac{2\pi x}{\lambda}$$

Thus it can be seen that the input impedances **of open and short-circuited** loss-free lines are always reactive **and vary from** $+j\infty$ **to** $-j\infty$ as the phase βx changes with either length or **frequency. For** a fixed frequency, figure 1.5 shows the variation **of reactance.**

Note that increasing the length of the line by **exactly half a** wavelength does not alter the input impedance. **Similar variations** *of input reactance occur if the frequency of the* ***input signal is*** varied and fed into a fixed length of line.

A loss-free line one-quarter wavelength long **terminated in a** load Z_L has an input impedance of

$$Z_{input} = Z_0 \left[\frac{Z_L + jZ_0 \tan \pi/2}{Z_0 + jZ_L \tan \pi/2}\right] \doteq \frac{Z_0^{\,2}}{Z_L}$$

Since $\beta = 2\pi/\lambda$

$$Z_0 \tan \frac{\pi}{2} >> Z_L \quad \text{and} \quad Z_L \tan \frac{\pi}{2} >> Z_0$$

This shows that a quarter wavelength line **may be used as an impedance-matching device.**