Semigroups of Linear Operators and Applications

JEROME A. GOLDSTEIN

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To my parents,
Henrietta and Morris Goldstein

Preface

Like Monsieur Jordan in *Le Bourgeois Gentilhamme*, who found to his great surprise that he had spoken prose all his life, mathematicians are becoming aware of the fact that they have used semigroups extensively if not consciously.

EINAR HILLE

It is difficult to tell when semigroup theory began. The concept was formulated and named in 1904, but in an 1887 paper, Giuseppe Peano [1][†] wrote the system of linear ordinary differential equations

$$du_1/dt = a_{11}u_1 + \cdots + a_{1n}u_n + f_1(t)$$

$$\vdots$$

$$du_n/dt = a_{n1}u_1 + \cdots + a_{nn}u_n + f_n(t)$$

in matrix form as $du/dt = Au + f$ and solved it using the explicit formula

$$u(t) = e^{tA}u(0) + \int_0^t e^{(t-s)A} f(s)\,ds,$$

where $e^{tA} = \sum_{k=0}^{\infty} t^k A^k/k!$. That is, he transformed a complicated problem in one dimension to a formally simple one in higher dimensions and used the ideas of one-variable calculus to solve it. That is the essence of this book.

The spectral theory of self-adjoint and normal operators on Hilbert space is based on the same idea. The notion of a self-adjoint operator is very special, and spectral theory enables one to take more-or-less arbitrary functions of it. In semigroup theory one only wants to take the exponential function of an operator, so one can work in much greater generality. This allows for the possibility of many surprising applications and the extension to a setting of nonlinear operators.

Mathematicians started taking one-parameter semigroup theory seriously in the 1930s. Perhaps its development became inspired when it was realized that the theory had immediate applications to partial differential equations, Markov processes, and ergodic theory. In 1948, Einar Hille published his monograph *Functional Analysis and Semi-Groups* in the American Mathematical Society Colloquium Series. The theory continued to develop rapidly in the fifties, thanks largely to Ralph Phillips; Hille's monograph then evolved into the Hille-Phillips book the same title, which makes substantial additions and deletions to the material in Hille's original book. The Hille-Phillips book, together with Part I of the three-volume series by Dunford and Schwartz, served as a Bible for my generation, the students of the sixties.

[†] I am indebted to Eugenio Sinestrari for this reference.

In 1970 I wrote a short set of lecture notes for a course on semigroups of linear operators at Tulane University. The most striking feature of those notes was how disjoint they were from Hille-Phillips. Post-1957 results (such as the Neveu-Trotter-Kato approximation theorem and the Chernoff product formula) played a prominent role, and simple transparent proofs had been found for many of the older results. The main flaw in those notes was that they did not begin to indicate the wide scope of applications of the simple theory. In 1972, I wrote another set of notes for an analogous course on nonlinear semigroups. Some years later I accepted Gian-Carlo Rota's suggestion to expand the lecture notes, linear and nonlinear, into a book, with a great emphasis on the applications.

This project turned out to be bigger and much more complicated and time consuming than I anticipated. The result is two volumes: the present one on linear theory and applications and a forthcoming nonlinear one.

The emphasis is on motivation, heuristics, and applications. It is hoped (and planned) that this work will be of use to graduate students and professionals in science and engineering as well as mathematics. All the main results are well-known, but several of the proofs are new. An effort was made to solve some nontrivial initial value problems for parabolic and hyperbolic differential equations without doing the hard work associated with elliptic theory. The reason is pedagogical; we wish to get across some of the main ideas involved in Cauchy problems for partial differential equations as an easy consequence of semigroup theory. Besides partial differential equations, other areas of application include mathematical physics (Feynman integrals, scattering theory, etc.), approximation theory, ergodic theory, potential theory, classical inequalities, fluid motion, and so on.

The exercises marked with an asterisk range from difficult to very difficult indeed. Some of them are research results which are incidental to the text but of sufficient interest to deserve to be stated. The bibliography requires some explanation. As a graduate student, I was very impressed with the large list of references in Dunford-Schwartz. It led to many enjoyable evenings of browsing in the library. I thought it would be useful to compile a complete list of references on the theory and applications of operator semigroups. I tried, but I have not succeeded because, as I painfully discovered, the literature is simply too vast for me to keep up with. Nevertheless, a large list was compiled. This list, which covers more than three hundred single-spaced type pages, is cited in the References at the end of the book as Goldstein [24]. To include all of the relevant references here would have made the book unnecessarily long and expensive. Thus many important articles have not been included. Nevertheless, the bibliography presented here contains a fairly substantial and representative sampling of the literature. This should help those readers interested in learning more about the theory and applications than the text presents.

We use the Halmos symbols iff for "if and only if" and ■ to signify the end of a proof.

Over the years, I have had various opportunities to lecture on the material in this book and in the forthcoming one. For their kind invitations or helpful comments or encouragement (or usually all three), I thank Geraldo Avila and

Djairo de Figueiredo (Brasília), Luiz Adauto Medeiros and Gustavo Perla-Menzala (Rio de Janeiro), Dick Duffin and Vic Mizel (Carnegie-Mellon), M. M. Rao (California-Riverside), David Edmunds and Eduard Fraenkel (Sussex), John Erdos (Kings-London), Rosanna Villella-Bressan (Padova), and my Tulane colleagues, Tom Beale, Ed Conway, Karl Hofmann, and Steve Rosencrans. I thank Brian Davies, Frank Neubrander, Simeon Reich, and Eric Schechter for correcting errors in the typescript. I thank Gian-Carlo Rota for his suggestion to write this book and for his encouragement. I record my admiration of Haim Brezis, the late Einar Hille, Tosio Kato, Peter Lax, Ralph Phillips, and Kôsaku Yosida for publishing such beautiful articles and for being constant sources of inspiration. I thank Susan Lam who typed the manuscript beautifully and efficiently. I gratefully acknowledge the partial support of the National Science Foundation. Finally, I thank my wife, Liz, and my children, David and Devra, for putting up with me and this project for all these years.

New Orleans
December, 1983

J. A. G.

Contents

Semigroups of Linear Operators and Applications

Chapter 0

A Heuristic Survey of the Theory and Applications of Semigroups of Operators

The evolution of a physical system in time is usually described by an initial value problem for a differential equation. (The differential equations can be ordinary or partial, and mixed initial value-boundary value problems are included.) The general setup is as follows. Let $u(t)$ describe the state of some physical system at time t. Suppose that the time rate of change of $u(t)$ is given by some function A of the state of the system $u(t)$. The initial data $u(0) = f$ is also given. Thus

$$\left.\begin{aligned} \frac{d}{dt} u(t) &= A[u(t)] \qquad (t \geq 0), \\ u(0) &= f. \end{aligned}\right\} \tag{0.1}$$

(For short $du/dt = Au$, $u(0) = f$.)

First of all we must make sense out of

$$du(t)/dt = \lim_{h \to 0} h^{-1}[u(t+h) - u(t)].$$

The function u takes values in a set $\mathscr{X}$. In order for $u(t+h) - u(t)$ to make sense, $\mathscr{X}$ is taken to be a vector space. In order that limits make sense in $\mathscr{X}$, $\mathscr{X}$ is taken to be a Banach space. (More generally, $\mathscr{X}$ could be a topological vector space or a differentiable manifold. But the desire to present a clean and complete theory with lots of applications in a reasonable number of pages led us to omit any setting more general than a Banach space.)

A is an operator (i.e. a function) from its domain $\mathscr{D}(A)$ in $\mathscr{X}$ to $\mathscr{X}$. The equation $du/dt = Au$ is interpreted to mean that $u(t)$ belongs to $\mathscr{D}(A)$ and that

$$\lim_{h \to 0} \|h^{-1}[u(t+h) - u(t)] - A[u(t)]\| = 0,$$

where $\|\cdot\|$ denotes the norm in $\mathscr{X}$.

Here are three examples.

Example 1. Let Ω be a bounded domain in n-dimensional Eudidean space $\mathbb{R}^n$ and let $\partial\Omega$ denote the (nice) boundary of Ω. Let $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ denote the Laplacian. Consider the following classical mixed initial-boundary value problem for the heat equation. We seek a function $w = w(t,x)$, defined for $0 \leq t < \infty$, $x \in \bar{\Omega} = \Omega \cup \partial\Omega$, such that

$$\left.\begin{aligned} \frac{\partial w}{\partial t} &= \Delta w && \text{for } (t,x) \in [0,\infty[\times \Omega, \\ w(0,x) &= f(x) && \text{for } x \in \Omega, \\ w(t,x) &= 0 && \text{for } x \in \partial\Omega, t \geq 0. \end{aligned}\right\} \tag{0.2}$$

(For consistency we should have $f(x) = 0$ for $x \in \partial\Omega$.) Write $u(t) = w(t,\cdot)$, regarded as a function of x, and take $\mathscr{X}$ to be a space of functions on Ω, e.g. $L^p(\Omega)$ for some $p \geq 1$ or $C(\bar{\Omega})$, the continuous functions on the closure of Ω. The derivatives du/dt and $\partial w/\partial t$ are both limits of the difference quotient $h^{-1}[w(t+h,x) - w(t,x)$, the first limit being in the sense of the norm of $\mathscr{X}$ and the second limit being a pointwise one. Even so, we can formally identify $\partial w/\partial t$ with du/dt. Clearly the functions denoted by f in both (0.1) and (0.2) can be identified with each other. To define A we take $\mathscr{X} = C(\bar{\Omega})$ for definiteness. Let $\mathscr{D}(A) = \{v \in C(\bar{\Omega}) : v$ is twice differentiable, $\Delta v \in C(\bar{\Omega})$, and $v(x) = 0$ for each $x \in \partial\Omega\}$. Define $Av = \Delta v$ for $v \in \mathscr{D}(A)$. Equations (0.2) are thus written in the form (0.1). Note that the boundary condition of (0.2) is absorbed into the domain of definition of the operator A and into the requirement that $u(t) \in \mathscr{D}(A)$ for all $t \geq 0$.

Example 2. Consider the initial value problem for the wave equation

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= \Delta w && \text{for } (t,x) \in [0,\infty[\, \times \mathbb{R}^n, \\ w(0,x) &= f_1(x) && \text{for } x \in \mathbb{R}^n, \\ \frac{\partial w}{\partial t}(0,x) &= f_2(x) && \text{for } x \in \mathbb{R}^n. \end{aligned} \tag{0.3}$$

For $\mathscr{X}$ we take a space of pairs of functions on $\mathbb{R}^n$. We set

$$u(t) = \begin{pmatrix} w(t,\cdot) \\ \dfrac{\partial w}{\partial t}(t,\cdot) \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$

i.e.

$$A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ \Delta v_1 \end{pmatrix}.$$

Then formally (0.3) becomes (0.1).

Example 3. Consider the initial value problem for the one-dimensional Hamilton-Jacobi equation

$$\left.\begin{aligned} \frac{\partial w}{\partial t} + F\left(\frac{\partial w}{\partial x}\right) &= 0 && (t \geq 0, x \in \mathbb{R}), \\ w(0,x) &= f(x) && (x \in \mathbb{R}). \end{aligned}\right\} \tag{0.4}$$

We take $\mathscr{X}$ to be a space of functions on $\mathbb{R}$ and set $u(t) = w(t,\cdot)$, $Av = -F(dv/dx)$. Then (0.4) formally becomes (0.1). Note that the operator A of this example is nonlinear, in contrast to the two preceding linear examples.

We return to the notion of a physical system which we imagine being housed in our (imaginary) experimental laboratory. In a *well-posed* physical experiment something happens, only one thing happens, and repeating the experiment with only small changes in the initial conditions or physical parameters produces only small changes in the outcome of the experiment. This suggests that if the initial value problem (0.1) is to correspond to a well-posed physical experiment, then we must establish an existence theorem, a uniqueness theorem, and a (stability) theorem which says that the solution depends continuously on the ingredients of the problem, namely the initial condition f and the operator A.

Suppose (0.1) is well-posed in the above (informal) sense. Let $T(t)$ map the solution $u(s)$ at time s to the solution $u(t + s)$ at time $t + s$. The assumption that A does not depend on time implies that $T(t)$ is independent of s; the physical meaning of this is that the underlying physical mechanism does not depend on time.

The solution $u(t + \tau)$ at time $t + \tau$ can be computed as $T(t + \tau)f$ or, alternatively, we can solve for $u(\tau) = T(\tau)f$, take this as initial data, and t units of time later the solution becomes $u(t + \tau) = T(t)(T(\tau)f)$. The uniqueness of the solution implies the semigroup property

$$T(t + \tau) = T(t)T(\tau) \qquad t,\tau > 0.$$

Also, $T(0) = I =$ the identity operator (this means that the initial condition is assumed), $t \to T(t)f$ is differentiable on $[0,\infty[$ [and $(d/dt)T(t)f = AT(t)f$ so that $u(t) = T(t)f$ solves (0.1)], and each $T(t)$ is a continuous operator on $\mathscr{X}$. (This reflects the continuous dependence of $u(t)$ on f.) The initial data f should belong to the domain of A, which is assumed to be dense in $\mathscr{X}$. Finally, each $T(t)$ is linear if A is linear.

We are thus led to the notion of a strongly continuous one-parameter semigroup of bounded linear operators on a Banach space $\mathscr{X}$. Such a semigroup is called a (C_0) *semigroup*; this terminology, introduced by Hille, has become standard. The definition is as follows. A family $T = \{T(t): 0 \le t < \infty\}$ of linear operators from $\mathscr{X}$ to $\mathscr{X}$ is called a (C_0) *semigroup* if

(i) $\|T(t)\| < \infty$ (i.e. $\sup\{\|T(t)f\|: f \in \mathscr{X}, \|f\| \le 1\} < \infty$) for each $t \ge 0$,
(ii) $T(t + s)f = T(t)T(s)f$ for all $f \in \mathscr{X}$ and all $t,s \ge 0$,
(iii) $T(0)f = f$ for all $f \in \mathscr{X}$,
(iv) $t \to T(t)f$ is continuous for $t \ge 0$ for each $f \in \mathscr{X}$.

T is called a (C_0) *contraction semigroup* if, in addition,

(v) $\|T(t)f\| \le \|f\|$ for all $t \ge 0$ and all $f \in \mathscr{X}$, i.e. $\|T(t)\| \le 1$ for each $t \ge 0$.

Roughly speaking, for most purposes it is enough to consider only (C_0) contraction semigroups. (This will be fully explained in Section 2 of Chapter I.)

Let T be a (C_0) semigroup. Define its *generator* (or *infinitesimal generator*) A by the equation

$$Af = \lim_{t \to 0} \frac{T(t)f - f}{t}.$$

where f is in the domain of A iff this limit exists. Formally, the semigroup property suggests that $T(t) =$ "e^{tA}" where $A = (d/dt)T(t)|_{t=0}$. This also suggests that the solution of (0.1) is given by $u(t) = T(t)f$, where T is the semigroup generated by A. The following result is basic.

THEOREM I (well-posedness theorem). *The initial value problem* (0.1) (*with A linear*) *is "well-posed" iff A is the generator of a* (C_0) *semigroup T. In this case the unique solution of* (0.1) *is given by* $u(t) = T(t)f$ *for f in the domain of A.*

See Chapter II, Theorem 1.2 and Exercise 1.5.4 for precise versions of this.

The obvious question that arises at this point is: which operators A generate (C_0) semigroups? For simplicity we work with a (C_0) contraction semigroup T. If A is the generator of T, think of $T(t)$ as e^{tA}. The formula

$$\frac{1}{\lambda - A} = \int_0^\infty e^{-\lambda t} e^{tA}\, dt,$$

which is valid when A is a number and $\lambda > \operatorname{Re}(A)$, suggests the operator version

$$(\lambda I - A)^{-1} f = \int_0^\infty e^{-\lambda t} T(t) f\, dt, \tag{0.5}$$

which turns out to be valid for all $\lambda > 0$ and all $f \in \mathscr{X}$; here I is the identity operator on $\mathscr{X}$. The f is there to make the integrand nice, namely, continuous and bounded in norm by the integrable function $\|f\|e^{-\lambda t}$. The estimate

$$\left\| \int_0^\infty e^{-\lambda t} T(t) f\, dt \right\| \le \int_0^\infty e^{-\lambda t} \| T(t) f \|\, dt$$
$$\le \int_0^\infty e^{-\lambda t} \|f\|\, dt = \|f\|/\lambda$$

suggests that:

$$\left.\begin{array}{l} \text{for each } \lambda > 0, \\ \lambda I - A \text{ maps the domain of } A \text{ onto } \mathscr{X} \\ \text{and } \|(\lambda I - A)^{-1} f\| \le \dfrac{1}{\lambda} \|f\| \text{ for all } f \in \mathscr{X}. \end{array}\right\} \tag{0.6}$$

THEOREM II (Hille-Yosida generation theorem). *A linear operator A generates a* (C_0) *contraction semigroup iff the domain of A is dense in* $\mathscr{X}$ *and* (0.6) *holds.*

One can recover the semigroup T from the generator A by inverting the Laplace transform (0.5) or by other methods, such as the formula

$$T(t)f = \lim_{n \to \infty} \left(I - \frac{t}{n} A \right)^{-n} f;$$

note that $(I - \alpha A)^{-1} = \lambda(\lambda I - A)^{-1}$ where $\lambda = 1/\alpha$.

The important implications in Theorems I and II are: (i) A densely defined operator A satisfying (0.6) generates a (C_0) semigroup. (ii) If A generates a (C_0) semigroup T, then the initial value problem (0.1) is well-posed and is governed by T. In other words, we solve (0.1) by solving equations of the form $\lambda h - Ah = g$ and getting a solution h satisfying the estimate $\|h\| \le \|g\|/\lambda$; this should be true for all $g \in \mathscr{X}$ and $\lambda > 0$.