Edwin H. Spanier

Algebraic Topology

Springer-Verlag
World Publishing Corporation

Edwin H. Spanier

Algebraic Topology

Springer-Verlag
World Publishing Corporation

Edwin H. Spanier
University of California
Department of Mathematics
Berkeley, Ca 94720

Library of Congress Cataloging in Publication Data

Spanier, Edwin Henry, 1921–
Algebraic topology.

Includes index.
1. Algebraic topology. I. Title.
QA612.S6 514'.2 81-18415
ISBN 0-387-90646-0 AACR2

Reprinted by World Publishing Corporation, Beijing, 1988
for distribution and sale in The People's Republic of China only

This book was originally published by McGraw-Hill, 1966.

Printed in the United States of America.
First Corrected Springer Edition.
9 8 7 6 5 4 3 2 1

ISBN 0-387-90646-0
ISBN 3-540-90646-0
ISBN 7-5062-0250-6

PREFACE

THIS BOOK IS AN EXPOSITION OF THE FUNDAMENTAL IDEAS OF ALGEBRAIC topology. It is intended to be used both as a text and as a reference. Particular emphasis has been placed on naturality, and the book might well have been titled *Functorial Topology.* The reader is not assumed to have prior knowledge of algebraic topology, but he is assumed to know something of general topology and algebra and to be mathematically sophisticated. Specific prerequisite material is briefly summarized in the Introduction.

Since *Algebraic Topology* is a text, the exposition in the earlier chapters is a good deal slower than in the later chapters. The reader is expected to develop facility for the subject as he progresses, and accordingly, the further he is in the book, the more he is called upon to fill in details of proofs. Because it is also intended as a reference, some attempt has been made to include basic concepts whether they are used in the book or not. As a result, there is more material than is usually given in courses on the subject.

The material is organized into three main parts, each part being made up of three chapters. Each chapter is broken into several sections which treat individual topics with some degree of thoroughness and are the basic organizational units of the text. In the first three chapters the underlying theme is the fundamental group. This is defined in Chapter One, applied in Chapter Two in the study of covering spaces, and described by means of generators and relations in Chapter Three, where polyhedra are introduced. The concept of functor and its applicability to topology are stressed here to motivate interest in the other functors of algebraic topology.

Chapters Four, Five, and Six are devoted to homology theory. Chapter Four contains the first definitions of homology, Chapter Five contains further algebraic concepts such as cohomology, cup products, and cohomology operations, and Chapter Six contains a study of topological manifolds. With each new concept introduced applications are presented to illustrate its utility.

The last three chapters study homotopy theory. Basic facts about homotopy groups are considered in Chapter Seven, applications to obstruction theory are presented in Chapter Eight, and some computations of homotopy groups of spheres are given in Chapter Nine. Main emphasis is on the application to geometry of the algebraic tools introduced earlier.

There is probably more material than can be covered in a year course. The core of a first course in algebraic topology is Chapter Four. This contains elementary facts about homology theory and some of its most important applications. A satisfactory one-semester first course for graduate students can be based on the first four chapters, either omitting or treating briefly Secs. 5 and 6 of Chapter One, Secs. 7 and 8 of Chapter Two, Sec. 8 of Chapter Three, and Sec. 8 of Chapter Four. A second one-semester course can be based on Chapters Five, Six, Seven, and Eight or on Chapters Five, Seven, Eight, and Nine. For students with knowledge of homology theory and related algebraic concepts a course in homotopy theory based on the last three chapters is quite feasible.

Each chapter is followed by a collection of exercises. These are grouped into sets, each set being devoted to a single topic or a few related topics. With few exceptions, none of the exercises is referred to in the body of the text or in the sequel. There are various types of exercises. Some are examples of the general theory developed in the preceding chapter, some treat special cases of general topics discussed later, and some are devoted to topics not discussed in the text at all. There are routine exercises as well as more difficult ones, the latter frequently with hints of how to attack them. Occasionally a topic related to material in the text is developed in a set of exercises devoted to it.

Examples in the text are usually presented with little or no indication of why they have the stated properties. This is true both of examples illustrating new concepts and of counterexamples. The verification that an example has the desired properties is left to the reader as an exercise.

The symbol ■ is used to denote the end of a proof. It is also used at the end of a statement whose proof has been given before the statement or which follows easily from previous results. Bibliographical references are by footnotes in the text. Items in each section and in each exercise set are numbered consecutively in a single list. References to items in a different section are by triples indicating, respectively, the chapter, the section or exercise set, and the number of the item in the section. Thus 3.2.2 is item 2 in Sec. 2 of Chapter Three (and 3.2 of the Introduction is item 2 in Sec. 3 of the Introduction).

The idea of writing this book originated with the existence of lecture notes based on two courses I gave at the University of Chicago in 1955. It is a pleasure to acknowledge here my indebtedness to the authors of those notes, Guido Weiss for notes of the first course, and Edward Halpern for notes of the second course. In the years since then, the subject has changed substantially and my plans for the book changed along with it, so that the present volume differs in many ways from the original notes.

The final manuscript and galley proofs were read by Per Holm. He made a number of useful suggestions which led to improvements in the text. For his comments and for his friendly encouragement at dark moments, I am sincerely grateful to him. The final manuscript was typed by Mrs. Ann Harrington and Mrs. Ollie Cullers, to both of whom I express my thanks for their patience and cooperation.

I thank the Air Force Office of Scientific Research for a grant enabling me to devote all my time during the academic year 1962–63 to work on this book. I also thank the National Science Foundation for supporting, over a period of years, my research activities some of which are discussed here.

Edwin H. Spanier

LIST OF SYMBOLS

CONTENTS

2 COVERING SPACES AND FIBRATIONS 60

3 POLYHEDRA 106

4 HOMOLOGY 154

5 PRODUCTS 210

6 GENERAL COHOMOLOGY THEORY AND DUALITY 284

7 HOMOTOPY THEORY 362

THE READER OF THIS BOOK IS ASSUMED TO HAVE A GRASP OF THE ELEMENTARY concepts of set theory, general topology, and algebra. Following are brief summaries of some concepts and results in these areas which are used in this book. Those listed explicitly are done so either because they may not be exactly standard or because they are of particular importance in the subsequent text.

I SET THEORY[1]

The terms "set," "family," and "collection" are synonyms, and the term "class" is reserved for an aggregate which is not assumed to be a set (for example, the class of all sets). If X is a set and $P(x)$ is a statement which is either true or false for each element $x \in X$, then

[1] As a general reference see P. R. Halmos, *Naïve Set Theory*, D. Van Nostrand Company, Inc., Princeton, N.J., 1960.

$$\{x \in X \mid P(x)\}$$

denotes the subset of X for which $P(x)$ is true.

If $J = \{j\}$ is a set and $\{A_j\}$ is a family of sets indexed by J, their *union* is denoted by $\cup A_j$ (or by $\cup_{j \in J} A_j$), their *intersection* is denoted by $\cap A_j$ (or by $\cap_{j \in J} A_j$), their *cartesian product* is denoted by $\times A_j$ (or by $\times_{j \in J} A_j$), and their *set sum* (sometimes called their *disjoint union*) is denoted by $\vee A_j$ (or by $\vee_{j \in J} A_j$) and is defined by $\vee A_j = \cup (j \times A_j)$. In case $J = \{1,2, \ldots ,n\}$, we also use the notation $A_1 \cup A_2 \cup \cdots \cup A_n$, $A_1 \cap A_2 \cap \cdots \cap A_n$, $A_1 \times A_2 \times \cdots \times A_n$, and $A_1 \vee A_2 \vee \cdots \vee A_n$, respectively, for the union, intersection, cartesian product, and set sum.

A *function* (or *map*) f from A to B is denoted by $f\colon A \to B$. The set of all functions from A to B is denoted by B^A. If $A' \subset A$, there is an *inclusion map* $i\colon A' \to A$, and we use the notation $i\colon A' \subset A$ to indicate that A' is a subset of A and i is the inclusion map. The inclusion map from a set A to itself is called the *identity map of* A and is denoted by 1_A. If $J' \subset J$, there is an inclusion map

$$i_{J'}\colon \bigvee_{j \in J'} A_j \subset \bigvee_{j \in J} A_j$$

An *equivalence relation* in a set A is a relation $\sim$ between elements of A which is *reflexive* (that is, $a \sim a$ for all $a \in A$), *symmetric* (that is, $a \sim a'$ implies $a' \sim a$ for $a, a' \in A$), and *transitive* (that is, $a \sim a'$ and $a' \sim a''$ imply $a \sim a''$ for $a, a', a'' \in A$). The *equivalence class* of $a \in A$ with respect to $\sim$ is the subset $\{a' \in A \mid a \sim a'\}$. The set of all equivalence classes of elements of A with respect to $\sim$ is denoted by $A/\sim$ and is called a *quotient set of* A. There is a *projection map* $A \to A/\sim$ which sends $a \in A$ to its equivalence class. If J' is a nonempty subset of J, there is also a *projection map*

$$p_{J'}\colon \bigtimes_{j \in J} A_j \to \bigtimes_{j \in J'} A_j$$

(which is a projection map in the sense above).

Given functions $f\colon A \to B$ and $g\colon B \to C$, their *composite* $g \circ f$ (also denoted by gf) is the function from A to C defined by $(g \circ f)(a) = g(f(a))$ for $a \in A$. If $A' \subset A$ and $f\colon A \to B$, the *restriction of* f *to* A' is the function $f \mid A'\colon A' \to B$ defined by $(f \mid A')(a') = f(a')$ for $a' \in A'$ (thus $f \mid A' = f \circ i$, where $i\colon A' \subset A$), and the function f is called an *extension of* $f \mid A'$ *to* A.

An *injection* (or *injective function*) is a function $f\colon A \to B$ such that $f(a_1) = f(a_2)$ implies $a_1 = a_2$ for $a_1, a_2 \in A$. A *surjection* (or *surjective function*) is a function $f\colon A \to B$ such that $b \in B$ implies that there is $a \in A$ with $f(a) = b$. A *bijection* (also called a *bijective function* or a *one-to-one correspondence*) is a function which is both injective and surjective.

A *partial order* in a set A is a relation $\leq$ between elements of A which is reflexive and transitive (note that it is not assumed that $a \leq a'$ and $a' \leq a$ imply $a = a'$). A *total order* (or *simple order*) in A is a partial order in A such that for $a, a' \in A$ either $a \leq a'$ or $a' \leq a$ and which is *antisymmetric* (that is, $a \leq a'$ and $a' \leq a$ imply $a = a'$). A *partially ordered set* is a set with a partial order, and a *totally ordered set* is a set with a total order.

1 ZORN'S LEMMA *A partially ordered set in which every simply ordered subset has an upper bound contains maximal elements.*

A *directed set* Λ is a set with a partial-order relation $\leq$ such that for $\alpha, \beta \in \Lambda$ there is $\gamma \in \Lambda$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$. A *direct system of sets* $\{A^\alpha, f_\alpha{}^\beta\}$ consists of a collection of sets $\{A^\alpha\}$ indexed by a directed set $\Lambda = \{\alpha\}$ and a collection of functions $f_\alpha{}^\beta: A^\alpha \to A^\beta$ for every pair $\alpha \leq \beta$ such that

(*a*) $f_\alpha{}^\alpha = 1_{A^\alpha}: A^\alpha \subset A^\alpha$ for all $\alpha \in \Lambda$
(*b*) $f_\alpha{}^\gamma = f_\beta{}^\gamma \circ f_\alpha{}^\beta: A^\alpha \to A^\gamma$ for $\alpha \leq \beta \leq \gamma$ in Λ

The *direct limit* of the direct system, denoted by $\lim_\to \{A^\alpha\}$, is the set of equivalence classes of $\vee A^\alpha$ with respect to the equivalence relation $a^\alpha \sim a^\beta$ if there is γ with $\alpha \leq \gamma$ and $\beta \leq \gamma$ such that $f_\alpha{}^\gamma a^\alpha = f_\beta{}^\gamma a^\beta$. For each α there is a map $i_\alpha: A^\alpha \to \lim_\to \{A^\alpha\}$, and if $\alpha \leq \beta$, then $i_\alpha = i_\beta \circ f_\alpha{}^\beta$.

2 *Given a direct system of sets $\{A^\alpha, f_\alpha{}^\beta\}$ and given a set B and for every $\alpha \in \Lambda$ a function $g_\alpha: A^\alpha \to B$ such that $g_\alpha = g_\beta \circ f_\alpha{}^\beta$ if $\alpha \leq \beta$, there is a unique map $g: \lim_\to \{A^\alpha\} \to B$ such that $g \circ i_\alpha = g_\alpha$ for all $\alpha \in \Lambda$.*

3 *With the same notation as in theorem 2, the map g is a bijection if and only if both the following hold:*

(*a*) $B = \cup g_\alpha(A^\alpha)$
(*b*) $g_\alpha(a^\alpha) = g_\beta(a^\beta)$ *if and only if there is γ with $\alpha \leq \gamma$ and $\beta \leq \gamma$ such that* $f_\alpha{}^\gamma(a^\alpha) = f_\beta{}^\gamma(a^\beta)$

Let $\{A_j\}$ be a collection of sets indexed by $J = \{j\}$. Let Λ be the collection of finite subsets of J and define $\alpha \leq \beta$ for $\alpha, \beta \in \Lambda$ if $\alpha \subset \beta$. Then Λ is a directed set and there is a direct system $\{A^\alpha\}$ defined by $A^\alpha = \vee_{j \in \alpha} A_j$, and if $\alpha \leq \beta$, then $f_\alpha{}^\beta: A^\alpha \to A^\beta$ is the injection map. Let $g_\alpha: A^\alpha \to \vee_{j \in J} A_j$ be the injection map.

4 *With the above notation, there is a bijection $g: \lim_\to \{A^\alpha\} \to \vee_{j \in J} A_j$ such that $g \circ i_\alpha = g_\alpha$ (that is, any set sum is the direct limit of its finite partial set sums).*

An *inverse system of sets* $\{A_\alpha, f_\alpha{}^\beta\}$ consists of a collection of sets $\{A_\alpha\}$ indexed by a directed set $\Lambda = \{\alpha\}$ and a collection of functions $f_\alpha{}^\beta: A_\beta \to A_\alpha$ for $\alpha \leq \beta$ such that

(*a*) $f_\alpha{}^\alpha = 1_{A_\alpha}: A_\alpha \subset A_\alpha$ for $\alpha \in \Lambda$
(*b*) $f_\alpha{}^\gamma = f_\alpha{}^\beta \circ f_\beta{}^\gamma: A_\gamma \to A_\alpha$ for $\alpha \leq \beta \leq \gamma$ in Λ

The *inverse limit* of the inverse system, denoted by $\lim_\leftarrow \{A_\alpha\}$, is the subset of $\times A_\alpha$ consisting of all points (a_α) such that if $\alpha \leq \beta$, then $a_\alpha = f_\alpha{}^\beta a_\beta$. For each α there is a map $p_\alpha: \lim_\leftarrow \{A_\alpha\} \to A_\alpha$, and if $\alpha \leq \beta$, then $p_\alpha = f_\alpha{}^\beta \circ p_\beta$.

5 *Given an inverse system of sets $\{A_\alpha, f_\alpha{}^\beta\}$ and given a set B and for every $\alpha \in \Lambda$ a function $g_\alpha: B \to A_\alpha$ such that $g_\alpha = f_\alpha{}^\beta \circ g_\beta$ if $\alpha \leq \beta$, there is a unique function $g: B \to \lim_\leftarrow \{A_\alpha\}$ such that $g_\alpha = p_\alpha \circ g$ for all $\alpha \in \Lambda$.*

6 *With the same notation as in theorem 5, the map g is a bijection if and only if both the following hold:*

(*a*) $g_\alpha(b) = g_\alpha(b')$ *for all* $\alpha \in \Lambda$ *implies* $b = b'$
(*b*) *Given* $(a_\alpha) \in \times A_\alpha$ *such that* $a_\alpha = f_\alpha{}^\beta a_\beta$ *if* $\alpha \leq \beta$, *there is* $b \in B$ *such that* $g_\alpha(b) = a_\alpha$ *for all* $\alpha \in \Lambda$

Let $\{A^j\}$ be a collection of sets indexed by $J = \{j\}$. Let Λ be the collection of finite nonempty subsets of J, and define $\alpha \leq \beta$ for $\alpha, \beta \in \Lambda$ if $\alpha \subset \beta$. Then Λ is a directed set and there is an inverse system $\{A_\alpha\}$ defined by $A_\alpha = \times_{j \in \alpha} A^j$, and if $\alpha \leq \beta$, $f_\alpha{}^\beta\colon A_\beta \to A_\alpha$ is the projection map. For each $\alpha \in \Lambda$ let $g_\alpha\colon \times_{j \in J} A^j \to A_\alpha$ be the projection map.

7 *With the above notation, there is a bijection* $g\colon \times_{j \in J} A^j \to \lim_{\leftarrow} \{A_\alpha\}$ *such that* $g_\alpha = p_\alpha \circ g$ *(that is, any cartesian product is the inverse limit of its finite partial cartesian products).*

2 GENERAL TOPOLOGY[1]

A topological space, also called a space, is not assumed to satisfy any separation axioms unless explicitly stated. Paracompact, normal, and regular spaces will always be assumed to be Hausdorff spaces. A continuous map from one topological space to another will also be called simply a map.

Given a set X and an indexed collection of topological spaces $\{X_j\}_{j \in J}$ and functions $f_j\colon X \to X_j$, the *topology induced on* X by the functions $\{f_j\}$ is the smallest or coarsest topology such that each f_j is continuous.

1 *The topology induced on* X *by functions* $\{f_j\colon X \to X_j\}$ *is characterized by the property that if* Y *is a topological space, a function* $g\colon Y \to X$ *is continuous if and only if* $f_j \circ g\colon Y \to X_j$ *is continuous for each* $j \in J$.

A *subspace* of a topological space X is a subset A of X topologized by the topology induced by the inclusion map $A \subset X$. A *discrete subset* of a topological space X is a subset such that every subset of it is closed in X. The *topological product* of an indexed collection of topological spaces $\{X_j\}_{j \in J}$ is the cartesian product $\times X_j$, given the topology induced by the projection maps $p_j\colon \times X_j \to X_j$ for $j \in J$. If $\{X_\alpha\}_{\alpha \in \Lambda}$ is an inverse system of topological spaces (that is, X_α is a topological space for $\alpha \in \Lambda$ and $f_\alpha{}^\beta\colon X_\beta \to X_\alpha$ is continuous for $\alpha \leq \beta$) their inverse limit $\lim_{\leftarrow} \{X_\alpha\}$ is given the topology induced by the functions $p_\alpha\colon \lim_{\leftarrow} \{X_\alpha\} \to X_\alpha$ for $\alpha \in \Lambda$.

Given a set X and an indexed collection of topological spaces $\{X_j\}_{j \in J}$ and functions $g_j\colon X_j \to X$, the *topology coinduced on* X by the functions $\{g_j\}$ is the largest or finest topology such that each g_j is continuous.

[1] As general references see J. L. Kelley, *General Topology*, D. Van Nostrand Company, Inc., Princeton, N.J., 1955, and S. T. Hu, *Elements of General Topology*, Holden-Day, Inc., San Francisco, 1964.

2 *The topology coinduced on X by functions $\{g_j: X_j \to X\}$ is characterized by the property that if Y is any topological space, a function $f: X \to Y$ is continuous if and only if $f \circ g_j: X_j \to Y$ is continuous for each $j \in J$.*

A *quotient space* of a topological space X is a quotient set X' of X topologized by the topology coinduced by the projection map $X \to X'$. If $A \subset X$, then X/A will denote the quotient space of X obtained by identifying all of A to a single point. The *topological sum* of an indexed collection of topological spaces $\{X_j\}_{j \in J}$ is the set sum $\vee X_j$, given the topology coinduced by the injection maps $i_j: X_j \to \vee X_j$ for $j \in J$. If $\{X^\alpha\}_{\alpha \in \Lambda}$ is a direct system of topological spaces (that is, X^α is a topological space for $\alpha \in \Lambda$ and $f_\alpha{}^\beta: X^\alpha \to X^\beta$ is continuous for $\alpha \leq \beta$) their direct limit $\lim_\to \{X^\alpha\}$ is given the topology coinduced by the functions $i_\alpha: X^\alpha \to \lim_\to \{X^\alpha\}$ for $\alpha \in \Lambda$.

Let $\mathcal{A} = \{A\}$ be a collection of subsets of a topological space X. X is said to have a *topology coherent with* $\mathcal{A}$ if the topology on X is coinduced from the subspaces $\{A\}$ by the inclusion maps $A \subset X$. (In the literature this topology is often called the *weak topology* with respect to $\mathcal{A}$.)

3 *A necessary and sufficient condition that X have a topology coherent with $\mathcal{A}$ is that a subset B of X be closed (or open) in X if and only if $B \cap A$ is closed (or open) in the subspace A for every $A \in \mathcal{A}$.*

4 *If $\mathcal{A}$ is an arbitrary open covering or a locally finite closed covering of X, then X has a topology coherent with $\mathcal{A}$.*

5 *Let X be a set and let $\{A_j\}$ be an indexed collection of topological spaces each contained in X and such that for each j and j', $A_j \cap A_{j'}$ is a closed (or open) subset of A_j and of $A_{j'}$ and the topology induced on $A_j \cap A_{j'}$ from A_j equals the topology induced on $A_j \cap A_{j'}$ from $A_{j'}$. Then the topology coinduced on X by the collection of inclusion maps $\{A_j \subset X\}$ is characterized by the properties that A_j is a closed (or open) subspace of X for each j and X has a topology coherent with the collection $\{A_j\}$.*

The topology on X in theorem 5 will be called *the topology coherent with* $\{A_j\}$. A *compactly generated space* is a Hausdorff space having a topology coherent with the collection of its compact subsets (this is the same as what is sometimes referred to as a Hausdorff *k-space*).

6 *A Hausdorff space which is either locally compact or satisfies the first axiom of countability is compactly generated.*

7 *If X is compactly generated and Y is a locally compact Hausdorff space, $X \times Y$ is compactly generated.*

If X and Y are topological spaces and $A \subset X$ and $B \subset Y$, then $\langle A;B \rangle$ denotes the set of continuous functions $f: X \to Y$ such that $f(A) \subset B$. Y^X denotes the space of continuous functions from X to Y, given the *compact-open topology* (which is the topology generated by the subbase $\{\langle K;U \rangle\}$, where K is a compact subset of X and U is an open subset of Y). If $A \subset X$

and $B \subset Y$, we use $(Y,B)^{(X,A)}$ to denote the subspace of Y^X of continuous functions $f\colon X \to Y$ such that $f(A) \subset B$. Let $E\colon Y^X \times X \to Y$ be the *evaluation map* defined by $E(f,x) = f(x)$. Given a function $g\colon Z \to Y^X$, the composite

$$Z \times X \xrightarrow{g \times 1} Y^X \times X \xrightarrow{E} Y$$

is a function from $Z \times X$ to Y.

8 THEOREM OF EXPONENTIAL CORRESPONDENCE *If X is a locally compact Hausdorff space and Y and Z are topological spaces, a map $g\colon Z \to Y^X$ is continuous if and only if $E \circ (g \times 1)\colon Z \times X \to Y$ is continuous.*

9 EXPONENTIAL LAW *If X is a locally compact Hausdorff space, Z is a Hausdorff space, and Y is a topological space, the function $\psi\colon (Y^X)^Z \to Y^{Z\times X}$ defined by $\psi(g) = E \circ (g \times 1)$ is a homeomorphism.*

10 *If X is a compact Hausdorff space and Y is metrized by a metric d, then Y^X is metrized by the metric d' defined by*

$$d'(f,g) = \sup \{d(f(x),g(x)) \mid x \in X\}$$

3 GROUP THEORY[1]

A homomorphism is called a *monomorphism, epimorphism, isomorphism,* respectively, if it is injective, surjective, bijective. If $\{G_j\}_{j \in J}$ is an indexed collection of groups, their *direct product* is the group structure on the cartesian product $\times G_j$ defined by $(g_j)(g_j') = (g_j g_j')$. If $\{G_\alpha\}$ is an inverse system of groups (that is, G_α is a group for each α and $f_\alpha{}^\beta\colon G_\beta \to G_\alpha$ is a homomorphism for $\alpha \leq \beta$), their inverse limit $\lim_{\leftarrow} \{G_\alpha\}$ (which is a set) is a subgroup of $\times G_\alpha$.

Let A be a subset of a group G. G is said to be *freely generated* by A and A is said to be a *free generating set* or *free basis* for G if, given any function $f\colon A \to H$, where H is a group, there exists a unique homomorphism $\varphi\colon G \to H$ which is an extension of f. A group is said to be *free* if it is freely generated by some subset. For any set A a *free group generated by A* is a group $F(A)$ containing A as a free generating set. Such groups $F(A)$ exist, and any two are canonically isomorphic.

1 *Any group is isomorphic to a quotient group of a free group.*

A *presentation* of a group G consists of a set A of *generators*, a set $B \subset F(A)$ of *relations*, and a function $f\colon A \to G$ such that the extension of f to a homomorphism $\varphi\colon F(A) \to G$ is an epimorphism whose kernel is the nor-

[1] As a general reference for elementary group theory see G. Birkhoff and S. MacLane, *A Survey of Modern Algebra,* The Macmillan Company, New York, 1953. For a discussion of free groups see R. H. Crowell and R. H. Fox, *Introduction to Knot Theory,* Ginn and Company, Boston, 1963.