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THEORY OF VIBRATIONS

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Preface

This book is intended to serve as a short, concise, analytical text for a first semester graduate course. It is based upon lectures given by the Author in the Graduate Division of Applied Mathematics, while visiting Professor at Brown University, Providence, R. I. Owing to limitation in length, practical applications, exercises, and additional matter given in the lectures, have had to be omitted. These, however, would be supplied by the lecturer in any case. It has been necessary to assume that the reader has an elementary working knowledge of Bessel functions, Fourier's integral theorem, and Operational Calculus. Such knowledge may *now* be regarded as a pre-requisite for the analytical study of vibrational problems. The Chapters proceed in logical sequence, and as far as possible (apart from starred sections, which are intended for a second reading), in order of analytical difficulty.

Symbols and abbreviations. In general these are standard, but heavy type has been used to signify 'per unit length', 'per unit area', and the moment of inertia of a disk, normal type representing that for a cross-section. The symbol $\supset$ has been used to signify the p-multiplied Laplace transform. It was introduced by the Author in 1938, and is now standard in France. Being made with one motion of the pen, it is much simpler than any other notation yet proposed. A slight modification of the symbol for use with the *ordinary* L.T. will be found in reference [13] p. 23.

$f(t) \supset \phi(p)$	means	that $\phi(p)$ is the p-multiplied Laplace transform of $f(t)$,
$\simeq$	means	is approximately equal to,
$\sim$	means	is analogous to,
[ ]	means	an item in the list of references,

b.c.	means	boundary condition,
c.f.	means	complementary function,
c-s	means	cross-sectional,
d.d.c.	means	dynamic deformation curve,
D.E.	means	differential equation,
e.m.f.	means	electromotive force,
K.E.	means	kinetic energy,
l.h.s.	means	left hand side,
L.T.	means	*p*-multiplied Laplace transform,
m.o.i.	means	moment of inertia,
p.d.(s)	means	potential difference(s),
p.i.	means	particular integral,
r.h.s.	means	right hand side.

N.W.M.

Contents

CHAPTER I

Linear Systems having One Degree of Freedom

1. INTRODUCTION Vibration is ubiquitous! It occurs in every phase of life. The human body cannot survive without the beating of the heart, while speech or any mode of transportation, even the act of walking, is associated with vibration. Most vibrations are complicated, e.g. those of an automobile on a rough road. By aid of mathematical analysis based upon suitable assumptions, complicated vibrations may be split up into simple types, just as a periodic function, an alternating current, or a continuous sound wave, may be analysed into its Fourier components. To obtain a solution adequate to account for the behaviour of an intricate system, often depends largely on the skill with which the basic simplifying assumptions are made. This is especially the case if numerical values computed from the analysis have to be compared with observed data. Otherwise the assumptions may be less rigorous, e.g. when a purely qualitative description of a physical phenomenon is needed.

A vibrational system is essentially one having mass and stiffness, or their analogs. Stiffness implies that alteration in the configuration due to an applied force is accompanied by a change in potential energy (strain). Tension in a string is equivalent to stiffness of a bar.

The electrical analogs of mass and stiffness are inductance and elastance (reciprocal of capacitance). Alternatively, compliance (reciprocal of stiffness) and capacitance are analogous. Theoretically it is expedient, in certain cases, to consider 'idealised' or pure masses and stiffnesses, or their analogs. The vibrating system is then said to be 'discrete' in type. Examples are a relatively heavy mass vibrating on a coil spring, or the balance wheel and hair spring of a watch. Such systems, in which the motion is specified by only *one* coordinate,* are said to have *one degree of freedom.* In §70 the velocity c of a disturbance along the spring is $(\mathbf{s}/\mathbf{m})^{1/2}$, so when the spring has stiffness only, the mass is zero and c infinite. Thus a force applied at one end of the spring is communicated to the other end instantaneously, so the spring moves in phase throughout. This result is approximated by a spring and a relatively large mass, but since no *actual* velocity can exceed that of light, the 'idealised' spring must be regarded as a convenient fiction!

When the mass and stiffness (or their analogs) are *distributed,* either uniformly or non-uniformly without a break, the system is said to be *continuous,* e.g. a violin string, the skin of a kettledrum. In a rigorous sense all systems are continuous, for each element of a discrete system may be set into vibration independently. The pendulum of a clock may be regarded as a discrete vibrational system, whose frequency is sub-sonic. But if we remove the bob and tap either it or the pendulum rod, audible vibrations ensue. Thus each element of the discrete system is itself a continuous system, but the lowest frequencies of the latter far exceed in value that of the (idealised) discrete system which they represent jointly in practice. Moreover, in general a discrete system is one composed of continuous elements, whose lowest free frequencies are much greater than those of the composite system. The same argument is applicable to electrical circuits comprising inductance, capacitance, and resistance. In practice, every coil has capacitance and resistance, and free electrical oscillations may occur if the resistance is not too high, i.e. the system is a continuous type. But by connecting a relatively large capacitance in parallel with the coil, the system may be regarded as discrete. The frequency of the combination is now a small fraction of the fundamental of the coil alone. The current is substantially in phase throughout the circuit, which has one degree of freedom, since only one coordinate (current) is needed to describe the (analogous) motion completely. Moreover, there is no such physical quantity as a *pure* mass, stiffness, inductance, capacitance, or resistance.

*x for the mass, and θ for the wheel.

But from an analytical viewpoint, provided certain conditions are satisfied, it is expedient to consider each of these to exist separately.

2. Simple mass-spring system Fig. 2.1A is a schematic diagram for a discrete mechanical system, where a mass m is fixed to a uniform helical spring s, whose other end is anchored. If m is displaced from its central or equilibrium position (when at rest), it will execute oscillations about that position. The motion is specified completely by one coordinate x, so the system has one degree of freedom.* To simplify the mathe-

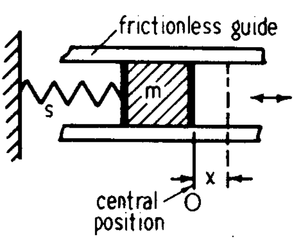

Fig. 2.1A

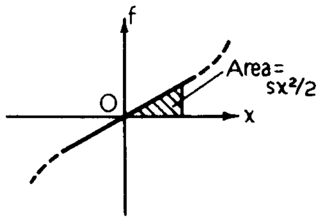

Fig. 2.1B.

matical analysis, we assume that (a) there is absence of loss, i.e. the motion is undamped, (b) the mass of the spring $m_s \ll m$. If the spring is loaded by known static forces, and the corresponding displacements measured, the graphical relationship between them is illustrated in Fig. 2.1B. This is the force-displacement 'characteristic' of the system, being linear provided the displacement is within certain limits. Beyond these, the graph takes the non-linear form indicated by the broken lines. We shall confine our attention to the linear part of the graph where the relationship is $f = sx$, s being the force per unit axial displacement ($\pm$), or the 'stiffness' of the *complete*

*A mass and spring without the guides shown in Fig. 2.1A has several degrees of freedom. If suspended vertically, it will oscillate like a pendulum, and vibrate axially too, etc.

spring of length l.† The work done in causing a displacement x is represented by the shaded area in Fig. 2.1B, which gives the strain or potential energy V stored in the spring. Thus

$$V = \int_0^x f\,dx = sx^2/2. \qquad 2.1$$

3. THE DIFFERENTIAL EQUATION By obtaining and then solving this, we can discuss the motion of the system. Since there is no driving or external force acting, the condition to be satisfied is that the sum of the internal forces must vanish. At any displacement $x \neq 0$, there are two forces, (a) sx the spring force tending to restore m to its central position 0, (b) the

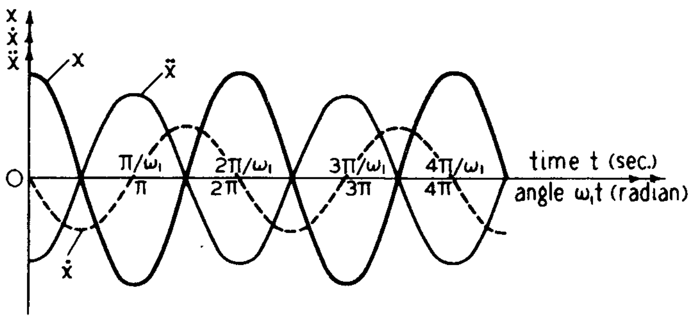

Fig. 3.1

inertial force md^2x/dt^2 by virtue of acceleration or deceleration, according as m is moving towards or away from 0. Thus

$$m\ddot{x} + sx = 0, \qquad 3.1$$

or

$$\ddot{x} + \omega_1^2 x = 0, \qquad 3.2$$

where $\omega_1 = (s/m)^{1/2}$. The complete solution of 3.2, with two arbitrary constants, is

†In general, stiffness $= df/dx$, which varies with x if the characteristic is non-linear. For *unit* length of spring, the stiffness in the linear case is sl.

$$x = A\cos\omega_1 t + B\sin\omega_1 t \qquad 3.3$$

$$= C\cos(\omega_1 t - \epsilon), \qquad 3.4$$

where $C = (A^2 + B^2)^{1/2}$, $\epsilon = \tan^{-1}(B/A)$.

To determine A, B, we have to specify 'initial' conditions, i.e. the displacement and velocity of m at $t = 0$.* Suppose the spring is extended by x_0 and released at $t = 0$. The initial conditions are $x = x_0$, $\dot{x} = 0$. Substituting the first into 3.3 gives $A = x_0$. Differentiating 3.3

$$\dot{x} = \omega_1(-A\sin\omega_1 t + B\cos\omega_1 t), \qquad 3.5$$

and for the condition $\dot{x} = 0$, $B = 0$. Inserting A, B, into 3.3 yields

$$x = x_0\cos\omega_1 t, \qquad 3.6$$

which gives the displacement of m from 0 at any time $t \geq 0$. The angular frequency of the motion is ω_1, the frequency in cycles per second $\omega_1/2\pi$, and the periodic time $2\pi/\omega_1$. By virtue of the cosinusoidal relationship, the motion of m is said to be *harmonic*. It is evident from 3.4 that whatever A, B, and, therefore, the initial conditions, the motion of m will be harmonic, since ϵ affects merely its phase.

By 3.6 the velocity of m is

$$\dot{x} = -\omega_1 x_0\sin\omega_1 t = \omega_1 x\cos(\omega_1 t + \pi/2), \qquad 3.7$$

and the acceleration

$$\ddot{x} = -\omega_1^2 x_0\cos\omega_1 t = \omega_1^2 x_0\cos(\omega_1 t + \pi). \qquad 3.8$$

By 3.6, 3.7, the phase of the velocity is $\pi/2$ in advance of the displacement, while by 3.6, 3.8, the acceleration is opposite to the displacement, i.e. π radians† in advance. The phase relationships are shown by the graphs in Fig. 3.1.

*The number of arbitrary constants and initial conditions is the same as the order of the differential equation.

†This is π/ω_1 seconds, since $(\omega_1 t + \pi) = \omega_1(t + \pi/\omega_1)$.

4. ENERGY EQUATION Writing $dx/dt = v$, we get $d^2x/dt^2 = (dv/dx)(dx/dt) = vdv/dx$. Substituting into 3.1 gives

$$mv\,dv/dx + sx = 0. \qquad 4.1$$

Multiplying throughout by dx and integrating, we obtain

$$m\int v\,dv + s\int x\,dx = C, \qquad \text{a constant}, \qquad 4.2$$

and the energy equation is

$$mv^2/2 + sx^2/2 = C. \qquad 4.3$$

This asserts that the sum of the kinetic energy of m, and the potential energy of s, is constant for all $|x| \leq x_0$, and $t \geq 0$. When $x = x_0$, $v = 0$, so

$$C = sx_0^2/2, \qquad 4.4$$

and when $x = 0$, $v = v_{max}$, so

$$C = mv_{max}^2/2, \qquad 4.5$$

these corresponding to the extreme and central positions, respectively.

Hence $$sx_0^2/2 = mv_{max}^2/2, \qquad 4.6$$

and the maximum potential (strain) and kinetic energies are equal. By 4.3, 4.4, the energy equation may be written

$$mv^2/2 = s(x_0^2 - x^2)/2. \qquad 4.7$$

5. ELECTRICAL ANALOG Referring to Fig. 5.1, suppose the capacitance has a charge Q, and at $t = 0$ the switch is closed.

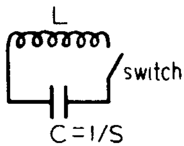

Fig. 5.1

Electrical oscillations will ensue. If I is the instantaneous current, since the sum of the p.d.s in the circuit must vanish, the D.E. is

$$L\,dI/dt + S\int I\,dt = 0, \qquad 5.1$$

where the elastance $S = 1/C$. Now $I = \dot{Q}$, so

$$L\ddot{Q} + SQ = 0. \qquad 5.2$$

This is identical in form with 3.1, and the mechanical and electrical systems are analogous, provided $x \backsim Q$, $m \backsim L$, $s \backsim S = 1/C$. Further, since $I = \dot{Q}$, it follows that $\dot{x} \backsim I$. Thus displacement is analogous to quantity of electricity, mass to inductance, stiffness to elastance (or compliance to capacitance), and velocity to current.

Hence by 3.6, 3.7, 4.3, 4.4,

$$Q = Q_0 \cos \omega_1 t, \qquad I = -\omega_1 Q_0 \sin \omega_1 t, \qquad 5.3$$

$$\omega_1 = (S/L)^{1/2} = 1/(LC)^{1/2}, \text{ and } LI^2/2 + SQ^2/2 = SQ_0^2/2. \qquad 5.4$$

Thus the sum of the electromagnetic and electrostatic energies in the system is constant, being independent of time. When $Q = 0$ the energy is wholly electromagnetic, and when $I = 0$, wholly electrostatic.

6. Mass suspended from spring This is illustrated in Fig. 6.1A. The weight or gravitational force $W = mg$ causes an extension $h = mg/s$, which fixes the equilibrium position, i.e. it in effect moves the origin from $x = 0$ to $x = h$. Accordingly in 3.2 we write $(x - h)$ for x, and obtain

$$\ddot{x} + \omega_1^2 x = \omega_1^2 h, \qquad 6.1$$

of which the complete solution with two arbitrary constants is

$$x = A \cos \omega_1 t + B \sin \omega_1 t + h, \qquad 6.2$$

or

$$(x - h) = A \cos \omega_1 t + B \sin \omega_1 t. \qquad 6.3$$

Hence motion of the mass in the systems of Figs. 2.1A, 6.1A about their respective equilibrium positions, is identical.

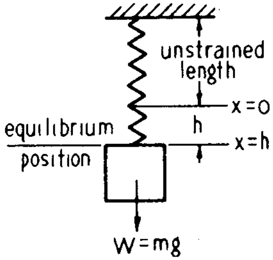

Fig. 6.1A

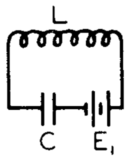

Fig. 6.1B

★*Electrical analog.* This is shown in Fig. 6.1B, where E_1 is a constant p.d. analogous to the force mg. The D.E. is

$$L\,dI/dt + S\int I\,dt = E_1 = SQ_1\,, \tag{6.4}$$

where Q_1 is the charge on C corresponding to a p.d. E_1. Then with $I = \dot{Q}$, 6.4 becomes

$$L\ddot{Q} + S(Q - Q_1) = 0, \tag{6.5}$$

or

$$\ddot{Q} + \omega_1^2 Q = \omega_1^2 Q_1\,. \tag{6.6}$$

Comparison with 6.1 shows that $h \sim Q_1$. The complete solution of 6.6 is, by 6.2

$$Q = A\cos\omega_1 t + B\sin\omega_1 t + Q_1\,; \tag{6.7}$$

also

$$I = \dot{Q} = \omega_1(-A\sin\omega_1 t + B\cos\omega_1 t). \tag{6.8}$$

If $Q = Q_0$, and $I = 0$ when $t = 0$, 6.7 gives $A = (Q_0 - Q_1)$, while from 6.8 $B = 0$. Using these values in 6.7, 6.8, yields

$$Q = (Q_0 - Q_1)\cos\omega_1 t, \tag{6.9}$$

and

$$I = -\omega_1(Q_0 - Q_1)\sin\omega_1 t. \tag{6.10}$$

7. Pneumatic stiffness Fig. 7.1 depicts a sealed enclosure of volume v_0 fitted with a short tube in which a *rigid* disk of area A and mass m_d can move freely. If displaced either inwards

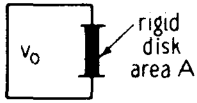

Fig. 7.1

or outwards, the disk will oscillate along its axis, in virtue of the 'stiffness' due to the enclosed air. Assuming adiabatic change, we have

$$pv^\gamma = C, \qquad \text{a constant,} \tag{7.1}$$

so $$d(pv^\gamma)/dx = \gamma p v^{\gamma-1} dv/dx + v^\gamma dp/dx = 0. \tag{7.2}$$

Multiplying by $Av^{-\gamma}$ gives

$$(\gamma pA/v)(dv/dx) = -A(dp/dx) = s, \tag{7.3}$$

the stiffness or force per unit displacement. The minus sign indicates that p increases with decrease in v, and vice-versa, in virtue of the negative slope of the adiabatic curve. Since $dv = Adx$, if p_0 is the static external air pressure, 7.3 gives

$$s = \gamma A^2 p_0/v_0 . \tag{7.4}$$

This is valid provided, (a) the displacement is such that the working arc of the 'characteristic' may be represented adequately by its tangent, (b) the internal pressure change is almost in phase everywhere, i.e. the wave length of sound $\gg$ the largest dimension of the enclosure.

During vibration, the mass of the disk is increased by virtue of the cyclically varying motion of the air in its neighborhood.* The additional mass is termed the 'accession to inertia' m_i [10]. For a disk radius a metre, as in Fig. 7.1, it is sensibly constant if $a\omega < 140$, but decreases with increase in ω thereafter. The

*Outside the enclosure.

total mass is $m_d + m_i = m$, so by §3, the natural frequency

$$\omega_1 = (s/m)^{1/2} = \{\gamma A^2 p_0/v_0(m_d + m_i)\}^{1/2}. \qquad 7.5$$

If the disk were suspended by a narrow annular surround of axial stiffness s_1, the total stiffness would be $s + s_1$, so

$$\omega_1 = \{(s + s_1)/m\}^{1/2}, \qquad 7.6$$

where m would be greater than in 7.5, owing to the mass of part of the annulus, and additional accession to inertia.

8. Reduced discrete system Systems of the type indicated in Fig. 8.1A, C having more than one spring, may be reduced

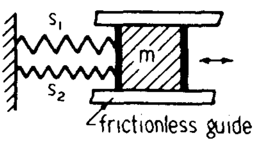

Fig. 8.1A

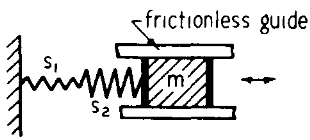

Fig. 8.1C

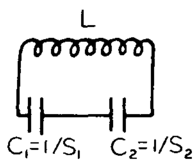

Fig. 8.1B

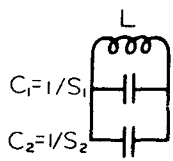

Fig. 8.1D

to an equivalent single mass-spring arrangement. In the language of electrical technology, the springs in Fig. 8.1A are in series, while those in Fig. 8.1C are in parallel. The combined stiffness in the first case is $s = (s_1 + s_2)$, which is analogous to elastances S_1, S_2, (Fig. 8.1B) in series. For the second case, neglecting the weight of the springs if m were suspended from them, a force f causes extensions $x_1 = f/s_1$, and $x_2 = f/s_2$, respectively. Thus the total extension is

$$x = x_1 + x_2 = f/s_1 + f/s_2 = f(1/s_1 + 1/s_2), \qquad 8.1$$

so the stiffness of the combination is

$$s = \text{force/extension} = 1/(1/s_1 + 1/s_2) = s_1 s_2/(s_1 + s_2), \qquad 8.2$$

and $s <$ either s_1 or s_2. This is analogous to the elastances S_1, S_2, (Fig. 8.1D) in parallel. The reduced system in either case, being described completely by one coordinate, has one degree of freedom, and the angular frequency is $\omega_1 = (s/m)^{1/2}$.

9. Lateral vibration of loaded uniform bar Referring to Fig. 9.1, we suppose the bar is vertical in its equilibrium

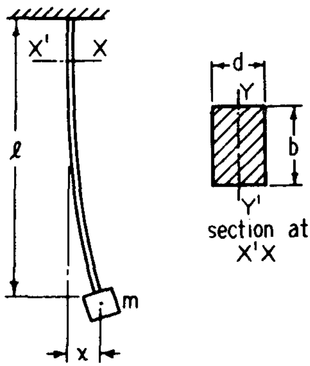

Fig. 9.1

position, and that its mass is negligible in comparison with that of m. The relationship between static horizontal force f and small displacement x is, neglecting the influence of the weight of the bar and that of m,

$$x = fl^3/3EI, \qquad 9.1$$

where I is the moment of inertia of the *section* of the bar about $Y'Y$, and E the modulus of elasticity. Since l^3/EI is constant,