DIMENSION THEORY

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PRINCETON

PRINCETON UNIVERSITY PRESS

LONDON: GEOFFREY CUMBERLEGE: OXFORD UNIVERSITY PRESS

1948

Revised edition, 1948
Third printing, 1952

Printed in the United States of America

Preface

In this book it has been the aim of the authors to give a connected and simple account of the most essential parts of dimension theory. Only those topics were chosen which are of interest to the general worker in mathematics as well as the specialist in topology.

Since the appearance of Karl Menger's well-known "Dimensionstheorie" in 1928, there have occurred important advances in the theory, both in content and method. These advances justify a new treatment, and in the present book great emphasis has been laid on the modern techniques of function spaces and mappings in spheres.

The algebraically minded reader will find in Chapter VIII a concise exposition of modern homology theory, with applications to dimension.

Historical references are made solely for the guidance of the beginning student, and no attempt has been made to attain completeness in this respect.

The authors wish to express their thanks to Drs. Samuel Eilenberg and William Flexner, who gave stimulating advice in preparing the manuscript, and to Mr. James Dugundji, who carefully read the proof and made valuable suggestions.

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October 1941

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DIMENSION THEORY

CHAPTER I

Introduction

1. The modern concept of dimension

"Of all the theorems of analysis situs, the most important is that which we express by saying that space has three dimensions. It is this proposition that we are about to consider, and we shall put the question in these terms: when we say that space has three dimensions, what do we mean? . . .

" . . . if to *divide* a continuum it suffices to consider as cuts a certain number of elements all distinguishable from one another, we say that this continuum is of *one dimension*; if, on the contrary, to divide a continuum it is necessary to consider as cuts a system of elements themselves forming one or several continua, we shall say that this continuum is of *several dimensions*.

"If to divide a continuum C, cuts which form one or several continua of one dimension suffice, we shall say that C is a continuum of *two dimensions*; if cuts which form one or several continua of at most two dimensions suffice, we shall say that C is a continuum of *three dimensions*; and so on.

"To justify this definition it is necessary to see whether it is in this way that geometers introduce the notion of three dimensions at the beginning of their works. Now, what do we see? Usually they begin by defining surfaces as the boundaries of solids or pieces of space, lines as the boundaries of surfaces, points as the boundaries of lines, and they state that the same proeedure can not be carried further.

"This is just the idea given above: to divide space, cuts that are called surfaces are necessary; to divide surfaces, cuts that are called lines are necessary; to divide lines, cuts that are called points are necessary; we can go no further and a point can not be divided, a point not being a continuum. Then lines, which can be divided by cuts which are not continua, will be continua of one dimension; surfaces, which can be divided by continuous cuts of one dimension, will be continua of two dimensions; and finally space, which can be divided by continuous cuts of two dimensions, will be a continuum of three dimensions."

These words were written by Poincaré in 1912, in the last year of his life. Writing in a philosophical journal,* Poincaré was concerned only with putting forth an intuitive concept of dimension and not an exact mathematical formulation. Poincaré had, however, penetrated very deep, in stressing the inductive nature of the geometric meaning of dimension and the possibility of disconnecting a space by subsets of lower dimension. One year later Brouwer† constructed on Poincaré's foundation a precise and topologically invariant definition of dimension, which for a very wide class of spaces (locally-connected separable metric) is equivalent to the one we use today.

Brouwer's paper remained almost unnoticed for several years. Then in 1922, independently of Brouwer, and of each other, Menger and Urysohn recreated Brouwer's concept, with important improvements; and what is more noteworthy, they justified the new concept by making it the cornerstone of an extremely beautiful and fruitful theory which brought unity and order to a large domain of geometry.

The definition of dimension we shall adopt in this book (see page 24) is due to Menger and Urysohn. In the formulation of Menger, it reads:

a) the empty set has dimension -1,

b) the dimension of a space is the least integer n for which every point has arbitrarily small neighborhoods whose boundaries have dimension less than n.

It is the opinion of the authors that none of the several other possible definitions of dimension has the immediate intuitive appeal of this one and none leads so elegantly to the existing theory.

2. Previous concepts of dimension

Before the advent of set theory mathematicians used dimension in only a vague sense. A configuration was said to be n-dimensional if the least number of real parameters needed to describe its points, in some unspecified way, was n. The dangers and inconsistencies in this approach were brought into clear view by two celebrated discoveries of the last part of the 19th century: Cantor's 1:1 correspondence between the points of a line and the points of a plane, and Peano's continuous mapping of an interval on the whole of a square. The first exploded the feeling that a plane is richer in points than a line, and showed that dimension can be changed by a 1:1 transformation. The second contradicted the belief that the dimension of a space could be defined as

* *Revue de métaphysique et de morale*, p. 486.

† Über den natürlichen Dimensionsbegriff, *Journ. f. Math.* 142 (1913), pp. 146–152.

the least number of continuous real parameters required to describe the space, and showed that dimension can be raised by a one-valued continuous transformation.

An extremely important question was left open (and not answered until 1911, by Brouwer): Is it possible to establish a correspondence between Euclidean n-space (the ordinary space of n real variables) and Euclidean m-space combining the features of both Cantor's and Peano's constructions, i.e. a correspondence which is both 1:1 *and* continuous? The question is crucial since the existence of a transformation of the stated type between Euclidean n-space and Euclidean m-space would signify that dimension (in the natural sense that Euclidean n-space has dimension n) has no topological meaning whatsoever! The class of topological transformations would in consequence be much too wide to be of any real geometric use.

3. Topological invariance of the dimension of Euclidean spaces

The first proof that Euclidean n-space and Euclidean m-space are not homeomorphic unless n equals m was given by Brouwer in his famous paper: Beweis der Invarianz der Dimensionenzahl, *Math. Ann.* 70 (1911), pp. 161–165. However, this proof did not explicitly reveal any simple topological property of Euclidean n-space distinguishing it from Euclidean m-space and responsible for the non-existence of a homeomorphism between the two. More penetrating, therefore, was Brouwer's procedure in 1913* when he introduced his "Dimensionsgrad," an integer-valued function of a space which was topologically invariant by its very definition. Brouwer showed that the "Dimensionsgrad" of Euclidean n-space is precisely n (and therefore deserved its name).

Meanwhile Lebesgue had approached in another way the proof that the dimension of a Euclidean space is topologically invariant. He had observed† that a square can be covered by arbitrarily small "bricks" in such a way that no point of the square is contained in more than three of these bricks; but that if the bricks are sufficiently small, at least three have a point in common. In a similar way a cube in Euclidean n-space can be decomposed into arbitrarily small bricks so that not more than $n + 1$ of these bricks meet. Lebesgue conjectured that this number $n + 1$ could not be reduced further, i.e. for any decomposition in sufficiently small bricks there must be a point common

* Über den natürlichen Dimensionsbegriff, loc. cit.

† "Sur la non applicabilité de deux domaines appartenant à des espaces de n et $n + p$ dimensions," *Math. Ann.* 70 (1911), pp. 166–168.

to at least $n + 1$ of the bricks. (The first proof of this theorem was given by Brouwer.*) Lebesgue's theorem also displays a topological property of Euclidean n-space distinguishing it from Euclidean m-space and therefore it also implies the topological invariance of the dimension of Euclidean spaces.

4. Dimension of general sets

The new concept of dimension, as we have already seen, gave a precise meaning to the statement that Euclidean n-space has dimension n, and thereby clarified considerably the entire structure of topology. Another feature which made the new dimension concept a milestone in geometry was the generality of the objects to which it could be applied. The lack of a precise definition of dimension, however unsatisfactory from an esthetic and methodological point of view, caused no real difficulty so long as geometry was confined to the study of relatively simple figures, such as polyhedra and manifolds. No doubt could arise, in each particular case, as to what dimension to assign to each of these figures. This situation changed radically, following the discoveries of Cantor, with the development of point-set theory. This new branch of mathematics tremendously enlarged the domain of what could be considered as "geometrical objects" and revealed configurations of complexity never before dreamt of. To associate with each of these objects a number which might reasonably be called a dimension was by no means a trivial task. What, for instance, was one to take as the dimension of the indecomposable continuum of Brouwer, or of Sierpiński's "curve" each of whose points is a ramification point?

Dimension-theory gives a complete answer to these questions. It assigns to every set of points in a Euclidean space (and even to every subset of Hilbert space†), no matter how "pathological," an integer which on intuitive and formal grounds strongly deserves to be called its dimension.

5. Different approaches to dimension

Before we proceed to a systematic study of dimension-theory let us pause to consider other possible ways of defining dimension.

We have already mentioned Lebesgue's method of proving the invariance theorem for the dimension of Euclidean spaces. His procedure can be used very well to establish a general concept of dimension: the Lebesgue dimension of a space is the least integer n with the

* Über den natürlichen Dimensionsbegriff, loc. cit.

† See Appendix for remarks concerning more general spaces.

property that the space may be decomposed into arbitrarily small domains not more than $n + 1$ of which meet. It turns out (see Chapter V) that this method of introducing dimension coincides with that due to Brouwer, Menger, and Urysohn. We shall now give other examples showing how topological investigations of very different nature all lead to the same concept of dimension.

A)* Let

$$f_i(x_1, \cdots, x_n), \qquad i = 1, \cdots, m \tag{1}$$

be m continuous real valued functions of n real unknowns, or what is the same, m continuous real-valued functions of a point in Euclidean n-space. It is one of the basic facts of analysis that the system of m equations in n unknowns,

$$f_i(x_1, \cdots, x_n) = 0, \tag{2}$$

has, in general, no solution if $m > n$. The words "in general" may be made precise as follows: by modifying the functions f_i very little one can obtain new continuous functions g_i such that the new system

$$g_i(x_1, \cdots, x_n) = 0 \tag{3}$$

has no solution. On the other hand, there do exist sets of n equations in n unknowns which are solvable, and which remain solvable after any sufficiently small modification of their left members. This property of Euclidean n-space can be made the basis of a general concept of dimension. A space X would be called n-dimensional if n is the largest integer for which there exist n continuous real-valued functions (1) defined over X such that the system of equations (2) has a solution which is essential in the sense explained above. It turns out that this "dimension" is again the same as the dimension of Brouwer, Menger, and Urysohn (see Chapter VI).

B) A modification of A) is this problem. Consider continuous transformations of a space in an n-sphere. Every point of the n-sphere may be regarded as a vector of length unity (a "direction") in Euclidean $(n + 1)$-space, so that instead of continuous transformations in the n-sphere, one may speak of continuous fields of non-vanishing $(n + 1)$-vectors. Suppose C is a closed set in a space X. Given a continuous field of non-vanishing $(n + 1)$-vectors on C, is it possible to extend this field, without changing the vectors on C, to a continuous field of non-

* The questions discussed in A), B), and C) are closely related to each other, and were raised by Alexandroff in a paper whose results underlie much of Chapters VI and VIII: Dimensionstheorie, *Math. Ann.* 106 (1932), pp. 161–238.

vanishing $(n + 1)$-vectors defined over all of X? It turns out that the dimension of X is the least integer n for which such an extension is possible for each closed set C and each continuous field of non-vanishing $(n + 1)$-vectors on C; in terms of mappings in the n-sphere, the least integer n with the property that any continuous mapping of any closed subset C into the n-sphere can be extended to all of X (see Chapter VI).

C) Another approach to dimension arises from homology theory. Consider 1-cycles (roughly, continuous closed curves) in a 2-dimensional manifold. Some of these bound a 2-dimensional part of the manifold, or in the notation of homology theory, are bounding cycles. On the other hand no 2-cycle (with the obvious exception of the vacuous 2-cycle all of whose coefficients are zero) can bound in the 2-manifold, because there is nothing 3-dimensional for it to bound. In a similar way an n-dimensional manifold contains non-vacuous bounding m-cycles for every m less than n, but contains only vacuous bounding n-cycles. Now, homology theory can be applied to arbitrary compact metric spaces. One may then define the "homology-dimension" of a compact metric space as the largest integer n for which there exist, with suitably chosen coefficients, non-vacuous bounding $(n - 1)$-cycles. The homology-dimension so defined turns out to be the same as our standard dimension (see Chapter VIII).

D) The intuitive perception of dimension associates with the word 1-dimensional, objects having length (or linear measure), with the word 2-dimensional, objects having area (or 2-dimensional measure), with the word 3-dimensional, objects having volume (or 3-dimensional measure), and so on. An attempt to make this intuitive feeling precise meets the obstacle that dimension is a topological concept while measure is a metrical concept. However, let us consider with a given metric space X all the metrics compatible with its topological structure. We find (see Chapter VII) that the dimension of X can be characterized as the largest real number p for which X, *in each metrization*, has positive p-dimensional Hausdorff measure.

6. Remarks

In this book we assume only a very elementary knowledge of point-set topology, such as is contained, for example, in the first chapters of Alexandroff-Hopf's *Topologie*, Julius Springer, Berlin, 1935; Kuratowski's *Topologie* I, Monografje Matematyczne, Warsaw, 1933;

Menger's *Dimensionstheorie*, B. G. Teubner, Leipzig, 1928. It will, however, be possible for the reader to refresh his acquaintance with the fundamentals of topology by the use of the index. The index contains, besides references to the definitions and results of dimension-theory, a considerable number of very brief discussions (and even some proofs) of topics in general topology which the development requires.

A large number of illustrative examples are included in the book, many of them without proof; these should be regarded as exercises.

Mathematical assertions of subsidiary importance are called Propositions. References to these are made according to the following scheme:

"By Proposition A)" means by Proposition A) of the same section and chapter in which the reference occurs.

"By Proposition 2 A)" means by Proposition A) of Section 2 of the same chapter in which the reference occurs.

"By Proposition III 2 A)" means by Proposition A) of Section 2 of Chapter III.

Throughout this book ***all spaces are separable metric,*** unless the contrary is explicitly stated. This limitation is made because there arise grave difficulties in extending dimension-theory to more general spaces. A brief discussion of some of these difficulties is given in the Appendix.

CHAPTER II

Dimension 0

Topology consists essentially in the study of the connectivity structure of spaces. The concept of a connected space, which in its present form is due to Hausdorff and Lennes, may be considered the root-concept from which is derived, directly or indirectly, the bulk of the important concepts of topology (homology or "algebraic connectivity" theory, local connectedness, dimension, etc.).

A space is *connected* if it cannot be split into two non-empty disjoint open sets. Equivalently: a space is connected if, except for the empty set and the whole space, there are no sets whose boundaries* are empty.

In this chapter we are concerned with spaces which are disconnected in an exceedingly strong sense, viz. have so many open sets whose boundaries are empty that every point may be enclosed in arbitrarily small sets of this type.

1. Definition of dimension 0

Definition II 1. A space X has *dimension* 0 *at a point* p if p has arbitrarily small neighborhoods† with empty boundaries, i.e. if for each neighborhood U of p there exists a neighborhood V of p such that

$$V \subset U,$$
$$\text{bdry } V = 0.$$

A non-empty space X has *dimension* 0, $\dim X = 0$, if X has dimension 0 at each of its points.

A) It is obvious that the property of being 0-dimensional, or of being 0-dimensional at a point p, is a topological invariant.

B) A 0-dimensional space can also be defined as a non-empty space in which there is a basis* for the open sets made up of sets which are at the same time open and closed.

Example II 1. Every non-empty finite or countable space X is

* See index. Observe that any set whose boundary is empty is both open and closed, and conversely.

† By a neighborhood of a point we mean any open set containing the point.

0-dimensional.* For given any neighborhood U of any point p let ρ be a positive real number such that the spherical neighborhood of radius ρ about p (the set of all points whose distance from p is less than ρ) is contained in U. Let $x_1, x_2, \cdots$ be an enumeration of X and $d(x_i, p)$ the distance from x_i to p. There exists a positive real number ρ' which is less than ρ and different from all the $d(x_i, p)$. The spherical neighborhood of radius ρ' about p is then contained in U and its boundary is empty. Hence X is 0-dimensional.

In particular the set $\mathcal{R}$ of rational real numbers is 0-dimensional.

Example II 2. The set $\mathcal{J}$ of irrational real numbers is 0-dimensional. For given any neighborhood U of an irrational point p there exist rational numbers ρ and σ such that $\rho < p < \sigma$ and the set V of irrational numbers between ρ and σ is contained in U. In the space $\mathcal{J}$ of irrationals V is open, and has an empty boundary because every irrational point which is a cluster-point† of V is between ρ and σ and hence belongs to V.

Example II 3. The Cantor discontinuum‡ $\mathcal{C}$ (the set of all real numbers expressible in the form $\sum_1^\infty a_n/3^n$ where $a_n = 0$ or 2) is 0-dimensional.

Example II 4. Any set of real numbers containing no interval is 0-dimensional.

Example II 5. The set $\mathcal{J}_2$ of points in the plane both of whose coordinates are irrational is 0-dimensional. For any such point is contained in arbitrarily small rectangles bounded by lines having rational intercepts with the coordinate axes and intersecting them at right angles, and the boundaries of such rectangles do not meet $\mathcal{J}_2$.

Example II 6. The set $\mathcal{R}_2^1$ of points in the plane exactly one of whose coordinates is rational is 0-dimensional. For any such point is contained in arbitrarily small rectangles bounded by lines having rational intercepts with the coordinate axes and intersecting them at 45°, and the boundaries of such rectangles do not meet $\mathcal{R}_2^1$.

Example II 7. The set $\mathcal{R}_n$ of points in Euclidean n-space§ E_n all of whose coordinates are rational is 0-dimensional. For $\mathcal{R}_n$ is countable.

Example II 8. The set $\mathcal{J}_n$ of points in E_n all of whose coordinates are irrational is 0-dimensional. This is a simple generalization of Example II 5.

Remark. Suppose $0 \leqq m \leqq n$. Denote by $\mathcal{R}_n^m$ the set of points in

* Do not forget that unless the contrary is explicitly stated all spaces considered in this book are separable metric.

† See index.

‡ See Hausdorff: *Mengenlehre*, de Gruyter, Berlin 2nd ed. 1927, p. 134.

§ See index.

E_n exactly m of whose coordinates are rational. In Examples II 7 and II 8 we have seen that $\mathcal{R}_n^n = \mathcal{R}_n$ and $\mathcal{R}_n^0 = \mathcal{J}_n$ are 0-dimensional. It is true (Example II 12) that $\mathcal{R}_n^m$ is 0-dimensional for each m and n, but the proof depends essentially on the "Sum Theorem for 0-dimensional Sets," Theorem II 2, and the simple proof of Example II 6 cannot be generalized.

EXAMPLE II 9. The set $\mathcal{R}_\omega'$ of points in the Hilbert* cube I_ω all of whose coordinates are rational is 0-dimensional. (This set is not countable.)

Suppose

$$a = (a_1, a_2, \cdots)$$

is an arbitrary point in I_ω and U is a neighborhood of a in I_ω. By taking n large enough and p_i, q_i sufficiently close to a_i, $p_i < a_i < q_i$, $i = 1, \cdots, n$, one gets a neighborhood of a contained in U consisting of the points

$$x = (x_1, x_2, \cdots)$$

of I_ω whose first n coordinates are restricted by

$$p_i < x_i < q_i \tag{1}$$

(and whose other coordinates are restricted only by

$$|x_i| \leqq 1/i, \tag{2}$$

((2) being, of course, always present in I_ω).†

Now, suppose $a \in \mathcal{R}_\omega'$. By taking p_i and q_i irrational we get a neighborhood V of a each of whose boundary points in I_ω has at least one

* See index.

† We have to show that for each $\epsilon > 0$ we can find an integer n and a positive real number δ such that if $q_i - p_i < \delta$ for $i \leqq n$ then all x satisfying (1) and (2) also satisfy

$$\left[\sum_{i=1}^{\infty} (x_i - a_i)^2\right]^{\frac{1}{2}} < \epsilon. \tag{3}$$

To do this choose n so that

$$\sum_{i=n}^{\infty} \frac{1}{i^2} < \frac{1}{8}\epsilon^2$$

and δ so that

$$n\delta^2 < \tfrac{1}{2}\epsilon^2.$$

If $q_i - p_i$ is less than δ for $i \leqq n$, we have for all x satisfying (1) and (2) that

$$\sum_{i=1}^{\infty} (x_i - a_i)^2 < n\delta^2 + \sum_{i=n}^{\infty} \left(\frac{2}{i}\right)^2 < \tfrac{1}{2}\epsilon^2 + \tfrac{1}{2}\epsilon^2 = \epsilon^2;$$

which proves (3).