AN INTRODUCTION TO APPLIED MATHEMATICS

BY

J. C. JAEGER

AND

A. M. STARFIELD

SECOND EDITION

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PREFACE TO THE SECOND EDITION

THE first edition of this book was an attempt to give an introduction to the basic mathematics needed in physics and engineering. Only a knowledge of the principles of statics and dynamics and of the calculus was assumed; other techniques were developed *ab initio*, but as rigorously as possible, and an attempt was made to develop mathematical skill through the solution of a large number of problems of technical significance. To this end, in addition to conventional statics and dynamics, the book covered much of what is now described as 'mathematical methods', including vector analysis, numerical analysis, ordinary and partial differential equations, special functions, Fourier series, and Fourier and Laplace Transforms. In dynamics it included a long chapter on mechanical vibrations and another on electric circuit theory. Boundary value problems were introduced through the theory of bending of beams.

This new edition has been prepared largely by Professor Starfield. It has been found that most of the old material is still needed. In some places the examples have been modernized, for example solid-state devices replace vacuum tubes. Other areas have been extended; for instance servomechanisms are discussed in terms of the transfer function, and the applications of Fourier transforms have been widened to include ideas in communication theory. Also, over the last few years, the application of mathematics to biological and economic problems has greatly increased, and discussion and examples relating to both of these topics have been added.

The great change over the past two decades has been the development of computer techniques which have not only enormously widened the range of problems which can be solved, but have changed habits of thought in the formulation of problems and demanded new approaches in their solution. The emphasis has to some extent changed from problems leading to differential equations to those involving difference equations or an algorithmic approach. Moreover, the numerical solution of differential equations involves their expression in terms of difference equations and creates a new concern with questions of accuracy and the stability of solution. In more complicated cases, such as systems of equations and the numerical solution of partial differential equations, matrices have become an essential computing tool.

To cope with these changes two new chapters have been added; one on difference equations and the numerical solution of differential equations, and the other on matrices. The last two chapters on partial differential equations have also been rewritten and extended and a number of examples have been added for numerical solution on a computer.

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I

MATHEMATICAL MODELS AND DIFFERENTIAL AND DIFFERENCE EQUATIONS

1. Introductory

THERE are three essential steps in the solution of a problem in applied mathematics. In the first step, the problem is stated in mathematical terms. This means that the relevant variables are identified and that mathematical relationships are established between them, either by using physical laws or empirical evidence, or by hypothesis. The second step consists of the solution of the mathematical relationships, either by standard mathematical techniques, or, if these prove intractable, by numerical methods with the aid of a computer. Finally, in the third step, the solution is expressed in a form which enables one to interpret it and draw physical conclusions from it.

Most of the problems with which we shall be concerned will lead to mathematical relationships involving either differential, or, occasionally, difference equations. The subsequent chapters of this book are therefore largely concerned with the solution and applications of these equations. In this chapter we shall use simple examples to show how differential and difference equations arise, and will lay the groundwork for both the mathematical and numerical methods of solving them.

2. Mathematical models

To illustrate the first step in the solution procedure, that is, the development of a mathematical model, we discuss the problem of predicting the growth of a population.

In the simplest case, this problem involves only two variables; the independent variable is the time t, and the dependent variable is the size of the population x. The mathematical relationship between x and t is then determined by the specific conditions relating to the population we are studying. For a large but uncrowded human population, with an unlimited food supply, we could argue that both the birth-rate and mortality rate at any time are proportional to the size of the population at that time,

$$\text{birth-rate} = bx$$

and

$$\text{mortality rate} = cx,$$

where b and c are constants. The rate of increase of the population is then

$$\frac{dx}{dt} = bx - cx, \tag{1}$$

which is an *ordinary differential equation*.

Alternatively, if we consider an insect population where one generation dies out before the next generation hatches, it is unsatisfactory to think of the population as a continuous function of time. We therefore introduce the notation $x_1, x_2, x_3,\ldots$, etc., for the size of generation $1, 2, 3,\ldots$, etc., and postulate that the increase in population from generation r to generation $r+1$ should be directly proportional to the size of the rth generation,

$$x_{r+1} - x_r = kx_r, \tag{2}$$

where k is a constant. This is an example of a *difference equation*.†

Both the previous models are deterministic in the sense that statistical fluctuations are ignored. If the population under study is sufficiently large, it is reasonable to assume that chance effects can be neglected. This is not true of small populations, where one must develop a probabilistic or stochastic model which includes an element of chance.

To take an extreme example, suppose that we start with a single cell at time $t = 0$, and that we wish to predict its subsequent division and subdivision. Suppose further that empirical evidence indicates that there is a probability q that the cell will divide in time T. We then cannot say how many cells we will have at time T; only that there is a probability q that we will

† Ex. 4 of § 112 illustrates some of the differences between a continuous (differential equation) and discrete (difference equation) formulation of the same problem.

have two cells and a probability $1-q$ that we will still have one cell. If we introduce the notation $p_{m,n}$ to denote the probability that we will have precisely m cells at time nT, then

$$\left.\begin{array}{r} p_{1,0} = 1, \quad p_{r,0} = 0 \quad \text{for } r \geqslant 2 \\ p_{1,1} = 1-q, \quad p_{2,1} = q, \quad p_{r,1} = 0 \quad \text{for } r \geqslant 3 \end{array}\right\}. \tag{3}$$

Using the theory of probability, we can then argue that at time $2T$

$$p_{1,2} = (1-q)^2, \quad p_{2,2} = q(1-q)(2-q), \quad p_{3,2} = 2q^2(1-q),$$
$$p_{4,2} = q^3, \quad p_{r,2} = 0 \quad \text{for } r \geqslant 5,$$

and so on. For example the argument leading to $p_{4,2}$ is that there can only be four cells at time $2T$ if the first cell divides at time T and both halves subsequently divide again. The probability of the former is q, of the latter q^2, and so the probability of both events occurring is q^3.

The probability distribution $p_{m,n}$ at each time nT gives a full statistical picture of the process of cell division. To compare this with a deterministic model, we calculate the average size or *expected value* of the population at nT. This is defined as

$$\bar{p}_n = \sum_{m=1}^{\infty} m p_{m,n}, \tag{4}$$

and it can be shown that, in this example,

$$\bar{p}_{n+1} = (1+q)\bar{p}_n. \tag{5}$$

Comparing (2) and (5) we see that, for this particular example, the deterministic model describes the average behaviour of the stochastic model.

Deterministic models will be perfectly adequate for most of the problems that we shall study. It is, however, important to bear in mind that some problems (see, for example, § 113 Ex. 1, and Ex. 6 at the end of Chapter XIV) can properly be described only by a stochastic model.

3. Solution and interpretation of results

The second step in the solution of a problem involves the actual solution of the mathematical model. In the example of the previous section, both the differential equation § 2 (1) and the difference equation § 2 (2) can be solved by the methods of Chapters III and XIV. The solution to the differential equation is

$$x(t) = Ae^{(b-c)t}, \tag{1}$$

where A is the size of the population at $t = 0$, while the solution to the difference equation is

$$x_r = x_1(1+k)^{r-1}, \tag{2}$$

which expresses subsequent generations in terms of the size of the first generation x_1.

We will not attempt to find the general form of the probabilities $p_{m,n}$ of § 2 (3). However, we notice that there is a definite argument which enables one to proceed from the probabilities at time nT to those at time $(n+1)T$, and this argument could be developed into a computer program which would calculate and print out tables of $p_{m,n}$ for different values of q.

Similarly, if we did not know how to solve the difference equation § 2 (2), we notice that writing it in the form

$$x_{r+1} = (1+k)x_r \tag{3}$$

suggests a direct method of calculating x_2 from x_1, x_3 from x_2, and so on, for a given value of k. This simple correspondence between difference equations and computer routines is one of the reasons why difference equations are important. In fact, in a more realistic model of population growth, k in (3) might well not be a constant but rather a complicated function of r and x_r, depending on environmental and other effects. It is then unlikely that a mathematical solution of (3) could be found, and the only way to solve the problem is to compute x_2, x_3,... etc., step by step.

The final step in the solution of a problem is the interpretation of the results. Apart from presenting the results as graphs or tables for various values of the constants, it is possible to draw some definite conclusions from solutions such as (1) and (2). For example, from (1) we see that the population will increase with time if $b > c$ (which we could have concluded from the differential equation § 2 (1) without even solving it). A less trivial result is that, if $b > c$, the population will double in a time period $(\ln 2)/(b-c)$† and will double again in a similar time period. In the case of the discrete model (2), the population doubles every $1+\{\ln 2/\ln(1+k)\}$ generations.

The preceding discussion indicates the importance of differential equations, difference equations, and computer methods in the solution of problems. In the next two sections we shall introduce the terminology of differential and difference equations, and discuss computer methods in greater detail.

4. Differential equations and difference equations. Definitions

Any relation between the independent‡ variable x, the dependent variable y, and its successive derivatives $dy/dx, d^2y/dx^2,\ldots$, is called an *ordinary*§ *differential equation*. The *order* of a differential equation is the order of the highest differential coefficient occurring in it. Thus, for example,

$$\frac{d^2y}{dx^2}+xy = 1, \tag{1}$$

$$\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^3+y = 0, \tag{2}$$

and

$$\left(\frac{d^2y}{dx^2}\right)^2+y\left(\frac{dy}{dx}\right)^3+y = 0, \tag{3}$$

are all second-order differential equations.

All the differential equations we shall need will contain only rational integral algebraic functions of the differential coefficients (fractional powers of x and y may sometimes occur), and in such cases the degree of the highest differential coefficient is called the *degree* of the equation. Thus (1) and (2) are both of the second order and the first degree, while (3) is of the second order and second degree.

† The notation $\ln x$ for $\log_e x$ will always be used.

‡ When discussing the theory of differential equations we shall take the independent variable to be x and the dependent variable y. In applications the symbols are determined by the problems. It is assumed throughout that y has derivatives of all the orders involved for all values of x.

§ If there are two or more independent variables, the equation is a partial differential equation; these will be discussed in Chapter XV. Until then the word 'ordinary' will usually be omitted.

By far the most important type of differential equation with which we shall be concerned is that in which all terms are of at most the first degree in y and its derivatives. This is called an ordinary *linear* differential equation, and its general form for the nth order is

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y = \phi(x). \quad (4)$$

The quantities $a_0(x), a_1(x),\ldots, a_n(x)$ are called the coefficients; if these are all constants, the equation is referred to as an ordinary linear differential equation *with constant coefficients*, otherwise it is a differential equation *with variable coefficients*.

Equation (4) is called an *inhomogeneous* equation to distinguish it from the corresponding equation with $\phi(x) = 0$,

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y = 0, \quad (5)$$

which is called a *homogeneous equation*.†

Equations (4) and (5) have fundamental properties which distinguish them from all other types of differential equations. Considering the homogeneous equation (5) first, suppose that y_1 and y_2 are two different solutions of it, so that

$$a_0(x)\frac{d^n y_1}{dx^n}+a_1(x)\frac{d^{n-1}y_1}{dx^{n-1}}+\ldots+a_n(x)y_1 = 0 \quad (6)$$

and

$$a_0(x)\frac{d^n y_2}{dx^n}+a_1(x)\frac{d^{n-1}y_2}{dx^{n-1}}+\ldots+a_n(x)y_2 = 0. \quad (7)$$

If c_1 and c_2 are constants, it follows, by adding c_1 times (6) to c_2 times (7), that $c_1 y_1+c_2 y_2$ also satisfies (5). That is, if we know two solutions of (5), any linear combination of these is also a solution. Similarly, if $y_1, y_2,\ldots, y_n$ are n different solutions of (5), the general linear combination

$$c_1 y_1+c_2 y_2+\ldots+c_n y_n,$$

where $c_1, c_2,\ldots, c_n$ are any constants, is also a solution. This result does not hold for the inhomogeneous equation (4).

† The term 'homogeneous' is also used in a different context for certain special types of differential equations; cf. § 25. These are not of much importance in applied mathematics and no confusion is likely to arise.

The important result for the inhomogeneous linear equation (4) is that if y_1 is a solution of it with a function $\phi_1(x)$ on the right-hand side, and y_2 a solution with $\phi_2(x)$ on the right-hand side, and so on, then $y = y_1+y_2+...+y_n$ satisfies (4) with $\phi(x) = \phi_1(x)+\phi_2(x)+...+\phi_n(x)$ on the right-hand side. This can be confirmed by adding the equations of type (4) for y_1 to y_n. In many applications we will find that $\phi(x)$ refers to the cause and y describes the effect. The above result thus implies that, *if the equations governing the problem are linear*, the effects of a number of superposed causes can be added. This is known as the Principle of Superposition.

Exactly the same terminology and results apply to difference equations. Any relation between the terms $y_r, y_{r+1},..., y_{r+n}$ of a sequence is called a *difference equation* of order n. The *linear* difference equation of order n is

$$a_0 y_{r+n}+a_1 y_{r+n-1}+...+a_{n-1} y_{r+1}+a_n y_r = \phi(r), \tag{8}$$

and if the coefficients $a_0, a_1,..., a_n$ are independent of r we call (8) a linear difference equation with constant coefficients. If $\phi(r) = 0$, (8) is homogeneous, otherwise it is inhomogeneous. It can easily be shown that the general linear combination of n different solutions of the linear homogeneous equation is also a solution of the homogeneous equation, and that the Principle of Superposition holds for linear inhomogeneous difference equations.

However, neither of these results hold for non-linear differential or difference equations. For example, if y_1 and y_2 both satisfy

$$\frac{d^2y}{dx^2}+y\frac{dy}{dx}+y = 0, \tag{9}$$

neither $c_1 y_1+c_2 y_2$ nor even $c_1 y_1$ satisfies (9) because of the non-linear term $y\,dy/dx$.

The distinction between linear and non-linear differential or difference equations is of fundamental importance. Broadly speaking, it will become apparent that, for both differential and difference equations, the solution of linear equations with constant coefficients is relatively straightforward; linear equations with variable coefficients are more difficult to solve, but special