$1.75

Modern Operational Calculus

with applications in technical mathematics

By N. W. McLachlan

REVISED EDITION

MODERN OPERATIONAL CALCULUS

WITH APPLICATIONS IN TECHNICAL MATHEMATICS

BY

N. W. McLACHLAN

D.Sc. (Engineering), London; Professor of Electrical Engineering, Emeritus, University of Illinois; Walker-Ames Professor of Electrical Engineering, University of Washington (1954)

DOVER PUBLICATIONS, INC.
NEW YORK NEW YORK

Published simultaneously in Canada by McClelland and Stewart, Ltd.
Published in the United Kingdom by Constable and Company Limited, 10 Orange Street, London W. C. 2.

This new Dover edition, first published in 1962, is a revised version of the work first published by Macmillan and Company in 1948.

Manufactured in the United States of America

Dover Publications, Inc.
180 Varick Street
New York 14, N. Y.

PREFACE TO DOVER EDITION

PRIOR to its being reprinted, the text has been revised. Additions, alterations and corrections have been made where necessary to effect improvement. The definition given at (1) §1·11 is that of the p-multiplied Laplace transform, which is used throughout the book. The reason for the presence of p outside the integral sign is given in §2·182.

LONDON, *June*, 1962. N.W.M.

PREFACE

THIS book is intended as an introduction to *Modern Operational Calculus* based upon the Laplace transform. It is written for post-graduate engineers and technologists. Nevertheless the purely mathematical part may be useful to advanced undergraduates in mathematics. The subject has attracted much attention during the past three decades, owing largely to its relevance to Heaviside's operational method. The Laplace transform of a function as defined by (1) § 1·11, is identical with what Heaviside called its operational form. The L.T. method of solving ordinary and partial linear differential equations is of comparatively recent date. Being logical and unambiguous, it is preferable to that of Heaviside, which may now be laid to rest in all its glory—such is the march of scientific progress!

A number of theorems or rules are established in Chapter II, the proofs usually being more complete than those given originally. Theorems 6, 6*a*, 9, and 14* are new. In Chapters III–VI the various theorems are used (*a*) to solve ordinary and partial linear differential equations, (*b*) to evaluate difficult integrals, (*c*) to obtain mathematical relationships and expansions, (*d*) to derive L.T.S. of various functions. Inclusion of a modern treatment of periodic impulses of finite and infinitesimal duration based upon complex integration seemed desirable. Accordingly Chapter VII is devoted to this subject, and it is a résumé of work I did during 1936–8. It is not intended for those unacquainted with complex integration, but it may prove an incentive to the study of this important branch of mathematics. The reader is advised to commence at Chapter I. No mathematical text is intended to be read from cover to cover like a novel. Accordingly the sections marked * may be omitted in a first reading, and reference made to them from time to time as the necessity arises.

* This theorem, discovered in 1940, was published in the *Mathematical Gazette* **30**, 85, 1946.

Some remarks on the ever-controversial topic of rigour may be appropriate. The text is perhaps more rigorous than is usual in a work on technical mathematics, and in this respect the technical reader is asked to consider carefully the following remarks, which are based on my own experience.

Engineers set a high standard of mechanical accuracy in limit gauges and jigs used for precision work and mass production methods. By manufacturing parts accurate to 10^{-4} inch, or even less, any individual spare, of the thousands turned out, will fit immediately into the intricate machine of which it is a component. High accuracy in machining eliminates trial and error, so a perfect fit is assured *a priori*. This being so, it is reasonable to ask engineers and technologists to accept a similar situation where mathematics is concerned. For the accuracy of the engineer is analogous to the rigour of the mathematician. Moreover, in this book the validity of operations like inverting the order of integration in a repeated integral, differentiation under the integral sign, term by term differentiation and integration of infinite series, are checked as they occur. In other words, the 'mathematical limit gauge' is used to test the analysis at various stages, so that ultimately the answer is correct without reliance upon flukes. Appendices are given so that the reader will know which 'mathematical limit gauges' are needed, when and how they should be applied.

In some technical mathematics the lack of reasonable rigour introduces uncertainty in the analysis. This, the inadequate time allotted at college, and the way in which the subject has been expounded, is largely responsible for the scepticism of engineers and engineering faculties. In the specifications and working drawings of a machine, dimensions, limits, materials, and processes of manufacture must be stated unambiguously, so that any engineer in, say, the antipodes, may understand the designer's intentions exactly. Why should this principle not apply in technical mathematics?

The problems in §§ 8·1–8·6 *should be regarded as an integral part of the book.* They contain important formulae and additional theorems, the proofs of which would have taken up too much space for inclusion in the text. Bessel functions [reference 11] have been used freely, since they occur so often in modern applied and technical mathematics. In fact the solutions of a large proportion of problems therein may be expressed in terms of exponential and Bessel functions. The reader is expected to have an elementary knowledge of the latter, e.g. Chapters I, II, and the early part of VII in reference [11].*

Mr. H. V. Lowry kindly read the manuscript, and I am much indebted to him for his valuable suggestions.

LONDON, *January*, 1941. N. W. M.

THE lapse of some seven years, between writing the MS. and its publication, is due entirely to war and immediate post-war conditions, which have caused such an upheaval in the printing trade. The war has emphasised, however, the necessity for enhanced facilities for the study of Technical Mathematics. There is no 'chair' of Technical Mathematics at any University in Great Britain. The time is ripe for distinct departments of Pure, Applied, and Technical Mathematics. The Professor of Technical Mathematics would have to be trained in both mathematics and technical matters, while industrial experience and the ability to impart knowledge would be essential.

It is appropriate to refer to the discussion on Technical Mathematics at the Mathematical Association in April 1945.† In general, those who lecture to undergraduate and post-graduate engineers expressed the view that in future this subject must play a much more important part in the curriculum than it has done hitherto. The ultimate purpose is to enable engineers and technologists to acquire a sound but broad knowledge of mathematics and its application to technical matters. Difficult mathematical problems requiring an intensive knowledge of certain branches of the subject should be handed over to specialists.

It may not be out of place to quote from the proposal I made at the above discussion, namely, '. . . a report on "The Teaching of Mathematics to Engineers" be drawn up by repre-

*Throughout the text the numbers in [ ] indicate the references on p. 212.

† *Mathematical Gazette*, **29**, 145, 1945.

sentatives of the Mathematical Association and members of the leading Engineering Institutions. . . .'

Prof. T. A. A. Broadbent and Mr. A. L. Meyers have read and criticised the proofs in detail. Mr. Meyers generously undertook the laborious task of checking the whole of the analysis and the problems in §§ 8·1–8·6. I have great pleasure in recording my appreciation of their help which has been invaluable.

N. W. M.

LONDON, *May*, 1947.

SYMBOLS

$R(\nu)$ signifies the real part of ν : μ and ν are usually unrestricted numbers, i.e. they may be real, imaginary or complex : if real they are generally non-integral : m, n are usually positive integers : r may take any integral value including zero, which it is *convenient* to regard as a positive integer.

$f(t) \supset \phi(p)$ signifies that $\phi(p)$ is the p-multiplied Laplace transform of $f(t)$, t being real $\geqslant 0$. $f(t) \rightarrow \Phi(p)$ signifies that $\Phi(p)$ is the ordinary Laplace transform of $f(t)$ as defined at (1) §2·144. Unless stated to the contrary, or the preceding sign is used, the Laplace transforms in the book are the p-multiplied type defined by (1) § 1·11. See the first footnote on p. 2.

$f(x, t)$ signifies a function of x and t, e.g. $e^{-x^2/t}$, $x^3t^{-1/2}$.

$\phi(p\,;\ h_1, h_2)$ signifies a function of p, h_1, h_2 (see § 1·11).

$f(t) \sim e^{-t^2}/t$ signifies that the r.h.s. is an asymptotic formula for $f(t)$ when t is large enough.

$|x|$ signifies the modulus of x, always real and positive.

$\sum_{r=0}^{\infty} |f_r(t)|$ signifies that the *moduli* of all functions in the series are to be summed.

The range $t \geqslant h$ signifies all values from $t=h$ to $t \to +\infty$, $h \geqslant 0$.

$h_1 \leqslant t \leqslant h$ signifies all values in any *closed* interval of t, and *includes* the end points h_1, h. A closed interval is *finite*.

$h_1 < t < h$ signifies all values in an open interval or range of t, excluding the end points h_1, h.

$f(t)$ is continuous in $h_1 \leqslant t \leqslant h$, signifies that the function is continuous for all values of t in the closed interval (h_1, h) : the continuity in a finite interval implies that $f(t)$ is bounded.

$t \neq 0$ signifies that t may *not* have the value zero.

$p \to +0$ and $p \to -0$ signify that p approaches zero from (*a*) the positive side, (*b*) the negative side.

$\simeq$ signifies ' approximately equal to '.

$\left.\begin{array}{l} f(t) = e^t \\ = 0 \end{array}\begin{array}{l} 0<t<h \\ t<0,\ t>h \end{array}\right\}$ signifies that $f(t)$ is equal to e^t in the range $t=(0, h)$ but excluding the end points; and is zero for all $t<0$ and $t>h$, being undefined at $t=0, h$.

Heavy type, **L**, **R**, **C**, **G**, signifies inductance, resistance, etc., of *unit length of cable.*

When t is near to or approaches some limiting value, the notation $f(t)=\mathbf{O}(t^\nu)$ means that $|f(t)|<Kt^\nu$, K being an absolute positive constant independent of variables or parameters.

$f(t)=\mathbf{O}(1)$ means that the function is bounded. When $t\to+\infty$, $(t^2+a^2)^2=\mathbf{O}(t^4)$; $t(e^{-t}+1)=\mathbf{O}(t)$; $t^2/(1+t^2)=\mathbf{O}(1)$ with bound unity. When $t\to 0$, $(t^2+a^2)=\mathbf{O}(1)$; $\sin t=\mathbf{O}(t)$.

The symbols for the various mathematical functions in the text are used in reference 13, where the functions are defined. This reference work contains an extensive list of p-multiplied Laplace transforms for functions which occur in pure, applied and technical mathematics.

Lerch's Theorem in § 1·16.

In the Laplace transform sense, if $f(0)\neq 0$ but finite, a discontinuity occurs. Nevertheless when considered in any interval including the origin, $f(t)$ itself may be continuous, e.g. $\cos t$, $J_0(t)$, where $f(0)=1$. Under these conditions, in a broad sense (*a*) § 1·16 is applicable, and (conventionally) $f(t)$ may be regarded as continuous in $t\geqslant 0$. Instances of this will be found in §§ 2·240, 3·12, 5·13, (*a*) in C_1 and C_2 in § 9 Appendix III.

Similarly the analysis in § 7·12 is valid if $f(t)$ is finitely discontinuous, in the sense intended in 1° § 1·15, at the ends of the finite interval (h_1, h_2).

FOREWORD

This is addressed to the technical reader whose mathematical training covers a more restricted field than that of the pure mathematician. The technical reader's experience is often limited to continuous functions. Herein it is necessary to consider functions which may be either finitely or infinitely discontinuous. Differentiable functions must be continuous,* whereas integrable functions can be either continuous, or finitely and (in some cases) infinitely discontinuous. Apart from discontinuities, functions must be single valued to avoid ambiguity. If $y^2=x$, $y=\pm x^{1/2}$ (unless $x=0$), and the positive branch is chosen usually. It must be appreciated that *infinity is not a number*, but a limit which exceeds *any* number we care to name, however large it may be. For the sake of brevity, 'infinity' (∞) is used frequently in such a way as to appear to be a number, e.g. the upper limit in an integral. The appropriate viewpoint is that the value of the integral is required as the upper limit $\rightarrow\infty$.

In enunciating a theorem, the conditions for its validity must be stated fully. Otherwise the theorem might be used in cases where it did not hold. Sometimes a theorem may hold under conditions less stringent than those given in the proof. If stated as mere formalities, the theorems in Chapter II become brief and simple. Bereft of the conditions for their validity, they are analytically incomplete and cannot be used with confidence. Omission of logical steps in analysis is just as serious a defect as absence of credits in a cash account. The answer to a problem is essential, but *its correctness is imperative*,

* There are, however, certain continuous functions (not contemplated herein) which are not differentiable. Throughout the text a continuous function means one which is differentiable. Functions of the type illustrated graphically in Figs. 22, 25(*b*) are continuous and differentiable *between* their points of finite discontinuity. They are piecewise continuous.

a condition which can be satisfied only by analysis in the appropriate detail. By perseverance in the early stages of the text, and frequent consultation of the appropriate appendices, the reader will find that the proper mental attitude is soon acquired.

CONTENTS

THE LAPLACE TRANSFORM

1·11. Definition. Consider the infinite integral

$$p\int_0^\infty e^{-pt}f(t)\,dt=\phi(p),\quad\text{.....................(1)}$$

p being a suitable parameter, either real or complex, while $f(t)$ is a single-valued* function integrable in every positive interval of t. t is real and $\geqslant 0$. This integral, but without the external p, was introduced into mathematical analysis by Laplace about the year 1779. $\phi(p)$, the function obtained by evaluating the integral, we *define* to be the *p-multiplied Laplace Transform* of $f(t)$. If $\phi(p)$ is given, $f(t)$ is said to be its inverse or interpretation in terms of the real variable t. When the range of integration in (1) is $t=(0, +\infty)$, ϕ is a function of p alone. If the range is $t=(h_1, h_2)$, $0\leqslant h_1<h_2$, ϕ is a function of p, h_1 and h_2. We then write

$$p\int_{h_1}^{h_2} e^{-pt}f(t)\,dt=\phi(p\,;\;h_1, h_2),\quad\text{..............(2)}$$

the r.h.s. being defined as the p-multiplied L.T. of $f(t)$ for the interval $t=(h_1, h_2)$. Integral (2) may be written in the form (1), for if

$$\left.\begin{aligned}F(t)&=f(t)\\&=0\end{aligned}\right\}\dagger\;\begin{aligned}&\text{when } 0\leqslant h_1<t<h_2,\\&t<h_1,\ t>h_2,\end{aligned}$$

then

$$p\int_0^\infty e^{-pt}F(t)\,dt=\phi(p\,;\;h_1, h_2).\quad\text{.................(3)}$$

Integral (1) is a particular case of (2) with $h_1=0$ and $h_2\to+\infty$. When p is real and >0,‡ (1), (2) may be interpreted

* The question of $f(t)$ being continuous is discussed later. Integrability of $f(t)$ in $t=(0, h)$ implies that of $e^{-pt}f(t)$, a point to be remembered in connection with enunciation of the theorems which follow, e.g. § 1·21.

† Using Heaviside's unit function (Appendix I), this may be written in the form $F(t)=f(t)\,[H(t-h_1)-H(t-h_2)]$ also. Fig. 20 illustrates the case where $f(t)=t^3$, $h_1=0$, $h_2=h$.

‡ $p>0$ implies the reality of p, but since p may be complex sometimes, this distinctive wording is used generally.

geometrically as the areas of the exponentially damped function $pf(t)$ between the limits $t=(0,+\infty)$, $t=(h_1, h_2)$, respectively.

The p-multiplied Laplace transform of a function, as defined by (1), is usually identical with what Heaviside called its *operational form*,* and the latter nomenclature is used frequently. By way of variation, some writers refer to $\phi(p)$ as the *image* of the *original* function $f(t)$. The most rational terminology seems to be that where $\phi(p)$ is regarded as the p-multiplied Laplace transform or L.T. of $f(t)$. The name Laplace must be appended, since there are other systems, where Fourier, Gauss, Hankel, Hilbert, Mellin, and Stieltjes transforms exist. Herein as elsewhere [12, 13], we shall employ the symbol $\Rightarrow$, namely, a heavy u lying on its side. Thus to signify that $\phi(p)$ is the p-multiplied L.T. of $f(t)$ we write

$$\left.\begin{array}{l} f(t) \Rightarrow \phi(p) \\ \text{or} \quad \phi(p) \Leftarrow f(t) \end{array}\right\}^{\dagger}, \qquad \text{(4)}$$

the closed end of the symbol pointing to the L.T. This is a very convenient and terse notation, the symbol being formed by a single stroke of the pen.

1·12. Example. Find the L.T. of t^ν, if $\nu=u+iv$ and $R(\nu)=u>-1$.

From (1) § 1·11
$$\phi(p)=p\int_0^\infty e^{-pt}t^\nu\,dt, \qquad \text{(1)}$$
$$=\Gamma(1+\nu)/p^\nu, \qquad \text{(2)}$$
or
$$t^\nu \Rightarrow \Gamma(1+\nu)/p^\nu, \qquad \text{(3)}$$

by [12, p. 75]. Integral (1) diverges at the lower limit, unless $R(\nu)>-1$, so when $R(\nu)\leqslant -1$ the function has no L.T. $\Gamma(1+\nu)=\int_0^\infty e^{-t}t^\nu dt$ is the gamma function introduced by Euler in 1729 [11, p. 177].

* The Laplace transform was defined originally without the external p, and in certain cases this is expedient (see § 2·144). There are several points in favour of the p-multiplied L.T.: (*a*) identity with Heaviside's operational forms, (*b*) the dimensional equivalence of $f(t)$ and $\phi(p)$ [see § 2·182], (*c*) the L.T. of a constant A is itself, and not A/p. Formulae on pp. 208–211, and in the references on p. 212 are p-multiplied L.T.S., as also are those in the text, unless stated otherwise.

† This notation refers to the L.T. for the interval $t=(0,\infty)$. For the interval $t=(h_1, h_2)$, see (2) § 1·11.

1·13. Integral equation for f(t). In this case the unknown function $f(t)$ occurs *under* the integral sign. Thus if $\phi(p)$ is known, but $f(t)$ is unknown, (1) § 1·11 is an integral equation for $f(t)$. Additional examples will be found in § 2·244; Chapters III, IV; 1–5, § 8·5.

If
$$p\int_0^\infty e^{-pt}f(t)\,dt=\Gamma(1+\nu)/p^\nu, \quad\text{.................(1)}$$
then by § 1·12, $f(t)=t^\nu$ is *a* solution, provided $R(\nu)>-1$ to ensure convergence.

1·14. Uniqueness. We now ask if the solution given in § 1·13 is unique, i.e. is it the only solution? In reference [9] Lerch showed that if $f(t)$ is continuous in $t\geqslant 0$, but in certain cases may be unbounded at $+\infty$, it is determined uniquely by $\phi(p)$. Now t^ν is continuous in $t\geqslant 0$, but unbounded at $+\infty$, if ν is real >0, or if $R(\nu)>0$, so with this proviso it is a unique solution of (1) § 1·13. Continuity, however, imposes an unnecessary restriction, since $f(t)$ is determined uniquely by $\phi(p)$ in the case of certain finitely discontinuous *integrable* functions, and certain infinitely discontinuous integrable functions. Before considering these functions in relation to (1) § 1·11, we shall introduce some definitions.

1·15. Discontinuous functions. A function may be discontinuous in several ways:

1°. Finitely discontinuous as in Figs. 13–17, 20–22, 25, 27–29 at the positions of the thin vertical lines. These are known as ordinary or simple discontinuities. The functions illustrated in Figs. 21 *d*, *e*, 22, 25 *b* may be regarded as periodic *piecewise* * continuous functions. They are integrable over a finite range $(0, t)$, and expressible in Fourier series by using the established procedure.

2°. Infinitely discontinuous like t^{-1}, $t^{-1/2}$ or $\log t$ at $t=0$, where each function has an 'infinity'.

* A 'piecewise' continuous function is continuous in stretches, devoid of infinities, and integrable in any finite range of t. It may $\to\infty$ with t, e.g. the 'staircase' function of Fig. 25*a*. A thin vertical line at a discontinuity is conventional, and is *not* part of the graph. Near a discontinuity a function is considered as t approaches from either side. In Fig. 13, $f(t)=0$ as $t\to-0$, $f(t)=E_0$ as $t\to+0$: *at* $t=0$ it is undefined.

3°. Oscillatorily discontinuous like $\sin (1/t)$ at the origin. As $t \to +0$, the function oscillates with constant amplitude, but the rate of oscillation $\to \infty$, i.e. the interval between consecutive zeros $\to 0$.

So far as L.T.S. are concerned, we shall confine our attention to the type of function in 1°, and those in 2° which fall in the category illustrated below. When

$$-1 < \nu < 0, \quad \text{or} \quad -1 < R(\nu) < 0,$$

t^ν has an infinity at the origin. Nevertheless integral (1) § 1·13 converges, and t^ν is a unique solution thereof. Other examples are $\log t$ and the Bessel functions $Y_0(t)$, $K_0(t)$, illustrated in Figs. 2, 3. As explained in § 1·211, convergence of (1) § 1·11 at the origin depends upon the 'order of infinity' of $f(t)$ as $t \to +0$ being less than unity. This condition is satisfied by $\log t$, $Y_0(t)$, $K_0(t)$, all of which are $O(\log t)$ when t is small and *positive*. Since

$$\int_0^h \log t \, dt = h(\log h - 1), \quad (h > 0), \quad \ldots\ldots\ldots\ldots\ldots(1)$$

the integrals

$$\int_0^h e^{-pt} \log t \, dt, \quad \int_0^h e^{-pt} Y_0(t) dt, \quad \int_0^h e^{-pt} K_0(t) dt, \quad (p > 0), \quad \ldots(2)$$

converge by comparison.

Most of the functions considered herein exist when $t < 0$, but from the L.T. viewpoint, we consider the range $t \geqslant 0$ only. If $f(t)$ is the function, then for L.T. purposes we define as follows:

$$F(t) = f(t) H(t), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where $H(t)$ is Heaviside's unit or step function treated in Appendix I. This definition is equivalent to

$$\left.\begin{array}{l} F(t) = f(t) \\ = 0 \end{array}\right\} \begin{array}{l} t > 0 \\ t < 0. \end{array} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

For convenience, however, we shall usually take $f(t)$ to signify $F(t)$ as so defined. Since $f(t) = 0$ when $t < 0$, if $f(0) = 0$, there is no discontinuity at the origin, e.g. Fig. 20. But if $f(0) \neq 0$, e.g. $\cos t$, $J_0(t)$, which have the value unity at $t = 0$, a finite discontinuity occurs (see Fig. 2 for $J_0(t)$).