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Introduction

The deep relationship between linear algebra and the arithmetical properties of polynomial rings is well understood, and a highlight is naturally Jordan's classification theorem for linear transformations on a finite-dimensional vector space. The methods and results of finite-dimensional linear algebra seldom extend to, or have analogues in, infinite-dimensional operator theory. Thus it is remarkable to have a class of operators whose properties are closely related with the arithmetic of the ring H^∞ of bounded analytic functions in the unit disc and for which a classification theorem is available, analogous to Jordan's classical result. Such a class is the class C_0, discovered by B. Sz.-Nagy and C. Foiaş in their work on canonical models for contraction operators on Hilbert space. A contraction operator belongs to this class if and only if the associated functional calculus on H^∞ has a nontrivial kernel. The class C_0 is the central object of study of this monograph, but we have included other related topics where it seemed appropriate. In an effort to make the book as self-contained as possible we give an introduction to the theory of dilations and functional models for contraction operators (see Chapters 1 and 5). While this introduction is adequate for our purposes, the reader familiar with the basic book [**6**] by Sz.-Nagy and Foiaş will be able to put the subject matter of this monograph in a greater perspective. Prerequisites for this book are a course in functional analysis (Rudin [**2**], for instance, will cover most of what we need) and an acquaintance with the theory of Hardy spaces in the unit disc (either Hoffman [**1**] or Duren [**1**] covers the required material). In addition, knowledge of the trace class of operators is needed in Chapter 6 (see, for example, Gohberg and Krein [**1**]).

Quite possibly, the class C_0 is the best understood class of nonnormal operators on a Hilbert space, even though there are still unsolved problems and unexplored avenues. Besides its intrinsic interest and direct applications, operators of class C_0 are very helpful as a source of inspiration, and in constructing examples and counterexamples in other branches of operator theory. Interestingly, the class C_0 also surfaces in certain problems of control and realization theory. It is hoped that this book will be interesting for operator theorists (present or to be), as well as those theoretical engineers who are interested in the applications of operator theory.

I tried to make this book more useful by including a number of exercises for each section. The numbering of theorems, propositions, etc. is conceived such as to make cross-references easy. For instance, Theorem 8.1.8 is in §1 of Chapter 8, and it is followed by relation (8.1.9) and Lemma 8.1.10. The first numeral is omitted for references within the same chapter. Each chapter begins with a description of the material to be covered. References to the literature and historical comments are kept to a minimum in the text. There is an appendix dedicated to these questions.

My teachers, colleagues, and friends Ciprian Foiaş, Carl Pearcy, Béla Sz.-Nagy, and Dan Voiculescu encouraged me at various times to write this book. Part of the book or earlier versions of some chapters were written while I was at the University of Michigan, the Massachusetts Institute of Technology, the Mathematical Sciences Research Institute, and Indiana University. Much of the material was presented in a seminar at the University of Michigan. I am grateful to all of these institutions for their hospitality and to some of them for help in typing the manuscript.

My wife Irina, with her exceptional talent and warmth, has been an inspiration for me during most of my mathematical life. Irina helped me get through difficult times and gave me determination and ambition when I lacked them. This book is dedicated to her memory.

Hari Bercovici

Contents

CHAPTER 1

An Introduction to Dilation Theory

Any contraction, i.e., operator of norm ≤ 1, on a Hilbert space has a unitary dilation. This is Sz.-Nagy's theorem, and it was the starting point of an important branch in operator theory. In this chapter we give the basic elements of dilation theory, which will help us enter the subject proper of the book in Chapter 2. In Section 1 we present Sz.-Nagy's dilation theorem mentioned above. As a consequence we deduce the decomposition of any contraction into a direct sum of unitary and completely nonunitary parts. We also give a proof of the commutant lifting theorem, which relates the commutant of a contraction with the commutants of its isometric and unitary dilations. Section 2 contains more detailed information about the minimal isometric dilation of an operator. It is shown that the completely nonunitary summand of an isometry is a unilateral shift, and conditions are given on an operator which ensure that its minimal isometric dilation is a unilateral shift. An important result concerns the absolute continuity (with respect to Lebesgue arclength measure on the unit circle) of the minimal unitary dilation. In Section 3 we discuss the notions of cyclic multiplicity, quasisimilarity, and quasiaffine transforms. The latter two notions are weak forms of similarity. The most important result (Theorem 3.7) relates an operator T, with small cyclic multiplicity, to a simpler operator. This result is the starting point of the classification theory of operators of class C_0.

1. Unitary dilations of contractions. Let T be a contraction on the Hilbert space $\mathscr{H}$. We will use the following notation:

$$(1.1)\qquad \begin{aligned} D_T &= (I - T^*T)^{1/2}, & D_{T^*} &= (I - TT^*)^{1/2},\\ \mathscr{D}_T &= (\text{ran } D_T)^-, & \mathscr{D}_{T^*} &= (\text{ran } D_{T^*})^-. \end{aligned}$$

The operator D_T is called the *defect operator* of T and $\mathscr{D}_T$ the *defect space.* Using the functional calculus for selfadjoint operators, it is easy to see that the obvious relation $T(I - T^*T) = (I - TT^*)T$ implies

$$(1.2)\qquad TD_T = D_{T^*}T.$$

In particular, we have $T\mathscr{D}_T \subset \mathscr{D}_{T^*}$.

Easier to understand among contractions are the isometric and unitary operators. Arbitrary contractions can be related to isometries using dilations. We recall that if $\mathscr{K}$ is a Hilbert space, $\mathscr{H} \subset \mathscr{K}$ is a subspace, $S \in \mathscr{L}(\mathscr{K})$, and $T \in \mathscr{L}(\mathscr{H})$, then S is a dilation of T (and T is a power-compression of S) provided that

$$T^n = P_{\mathscr{H}} S^n \mid \mathscr{H}, \quad n = 0, 1, 2, \ldots.$$

If, in addition, S is an isometry (unitary operator) then S will be called an isometric (unitary) dilation of T. An isometric (unitary) dilation S of T is said to be minimal if no restriction of S to an invariant subspace is an isometric (unitary) dilation of T. The following result is left as an exercise.

1.3. LEMMA. *Let S be an isometric (unitary) dilation of T. Then S is a minimal isometric (unitary) dilation of T if and only if $\bigvee_{n=0}^{\infty} S^n\mathscr{H} = \mathscr{K}$ ($\bigvee_{n=-\infty}^{\infty} S^n\mathscr{H} = \mathscr{K}$).*

The proof of the next result is motivated by the following calculation:

$$\|x\|^2 - \|Tx\|^2 = (x, x) - (T^*Tx, x) = \|D_T x\|^2, \quad x \in \mathscr{H},$$

which shows that the operator $X : \mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$ defined by $Xx = Tx \oplus D_T x$ is isometric. Of course, X is not a dilation of T because it acts between two different Hilbert spaces.

1.4. THEOREM. *Every contraction $T \in \mathscr{L}(\mathscr{H})$ has a minimal isometric dilation. This dilation is unique in the following sense: if $S \in \mathscr{L}(\mathscr{K})$ and $S' \in \mathscr{L}(\mathscr{K}')$ are two minimal isometric dilations for T, then there exists an isometry U of $\mathscr{K}$ onto $\mathscr{K}'$ such that $Ux = x$, $x \in \mathscr{H}$, and $S'U = US$.*

PROOF. We first prove the uniqueness part. Thus, let $S \in \mathscr{L}(\mathscr{K})$ and $S' \in \mathscr{L}(\mathscr{K}')$ be two minimal isometric dilations of T and note that, by Lemma 1.3, we have

$$\mathscr{K} = \bigvee_{n=0}^{\infty} S^n \mathscr{H}, \qquad \mathscr{K}' = \bigvee_{n=0}^{\infty} S'^n \mathscr{H}.$$

If $\{x_j\}_{j=0}^{\infty}$ is a finitely nonzero family of vectors in $\mathscr{H}$, we have

$$\left\|\sum_{j=0}^{\infty} S^j x_j\right\|^2 = \sum_{j,k=0}^{\infty} (S^j x_j, S^k x_k).$$

Since S is an isometry, we have $(S^j x_j, S^k x_k) = (S^{j'} x_j, S^{k'} x_k)$ if $k - j = k' - j'$ and therefore

$$\begin{aligned}
\left\|\sum_{j=0}^{\infty} S^j x_j\right\|^2 &= \sum_{j\geq k}(S^{j-k}x_j, x_k) + \sum_{j<k}(x_j, S^{k-j}x_k) \\
&= \sum_{j\geq k}(S^{j-k}x_j, P_{\mathscr{H}}x_k) + \sum_{j<k}(P_{\mathscr{H}}x_j, S^{k-j}x_k) \\
&= \sum_{j\geq k}(P_{\mathscr{H}}S^{j-k}x_j, x_k) + \sum_{j<k}(x_j, P_{\mathscr{H}}S^{k-j}x_k) \\
&= \sum_{j\geq k}(T^{j-k}x_j, x_k) + \sum_{j<k}(x_j, T^{k-j}x_k),
\end{aligned}$$

where we used the fact that S is a power-dilation of T. A similar computation for S' shows that $\|\sum_{j=0}^{\infty} S^j x_j\| = \|\sum_{j=0}^{\infty} S'^j x_j\|$. This easily implies the existence of an isometry U of $\mathscr{K}$ onto $\mathscr{K}'$ satisfying

$$U\left(\sum_{j=0}^{\infty} S^j x_j\right) = \sum_{j=0}^{\infty} S'^j x_j$$

for every finitely nonzero sequence $\{x_j\}_{j=0}^{\infty}$ in $\mathscr{H}$. Clearly then $Ux = x$, $x \in \mathscr{H}$, and $S'U = US$, so that uniqueness is proved.

For the existence part, we define the space $\mathscr{K}_+$ by

$$\mathscr{K}_+ = \mathscr{H} \oplus \left(\bigoplus_{n=0}^{\infty} \mathscr{D}_n\right), \qquad \mathscr{D}_n = \mathscr{D}_T,\ n = 0, 1, 2, \ldots,$$

and the operator $U_+ \in \mathscr{L}(\mathscr{K}_+)$ by

$$U_+\left(x \oplus \left(\bigoplus_{n=0}^{\infty} d_n\right)\right) = Tx \oplus \left(\bigoplus_{n=0}^{\infty} e_n\right)$$

where $e_0 = D_T x$ and $e_n = d_{n-1}$, $n \geq 1$. Since $\|Tx\|^2 + \|D_T x\|^2 = \|x\|^2$, it is obvious that U_+ is an isometry. It is also clear that U_+ is an isometric dilation of T, if we identify the vector $x \in \mathscr{H}$ with the vector $x \oplus (\bigoplus_{n=0}^{\infty} 0) \in \mathscr{K}$; in fact $\mathscr{H}$ is invariant under U_+^* and $T^* = U_+^* \mid \mathscr{H}$. It remains to be shown that U_+ is minimal. It is clear that $\mathscr{H} \vee U_+\mathscr{H}$ contains all elements of the form $0 \oplus D_T x \oplus 0 \oplus \cdots$, $x \in \mathscr{H}$, so that

$$\mathscr{H} \vee U_+\mathscr{H} = \mathscr{H} \oplus \mathscr{D}_T \oplus \{0\} \oplus \cdots .$$

It now follows from the definition of U_+ that

$$\bigvee_{j=0}^{n} U_+^j \mathscr{H} = \mathscr{H} \oplus \underbrace{\mathscr{D}_T \oplus \mathscr{D}_T \oplus \cdots \oplus \mathscr{D}_T}_{n \text{ times}} \oplus \{0\} \oplus \cdots$$

and the minimality of U_+ follows from Lemma 1.3.

The following result is the counterpart of Theorem 1.4 for unitary dilations.

1.5. THEOREM. *Every contraction $T \in \mathscr{L}(\mathscr{H})$ has a minimal unitary dilation, unique in the sense specified in Theorem* 1.4.

PROOF. The uniqueness is proved using the same calculation as in the proof of Theorem 1.4, except that one would consider sums of the form $\sum_{j=-\infty}^{\infty} S^j x_j$, $x_j \in \mathscr{H}$. In order to prove the existence of a minimal unitary dilation we consider the space $\mathscr{K}$ defined as

$$\mathscr{K} = \left(\bigoplus_{j=-\infty}^{0} \mathscr{E}_j\right) \oplus \mathscr{H} \oplus \left(\bigoplus_{j=0}^{\infty} \mathscr{D}_j\right),$$

where $\mathscr{E}_{-j} = \mathscr{D}_{T^*}$ and $\mathscr{D}_j = \mathscr{D}_T$, $j = 0, 1, 2, \ldots$, and the operator $U \in \mathscr{L}(\mathscr{K})$ defined by

$$U\left(\left(\bigoplus_{j=-\infty}^{0} e_j\right) \oplus x \oplus \left(\bigoplus_{j=0}^{\infty} d_j\right)\right) = \left(\bigoplus_{j=-\infty}^{0} e'_j\right) \oplus x' \oplus \left(\bigoplus_{j=0}^{\infty} d'_j\right),$$

where $x' = Tx + D_{T^*}e_0$, $d'_0 = -T^*e_0 + D_T x$, $d'_j = d_{j-1}$, $j \geq 1$, and $e'_j = e_{j-1}$, $j \leq 0$. The space $\mathscr{K}_+$, constructed in the previous proof, can be identified with $\{0\} \oplus \mathscr{K}_+ \subset \mathscr{K}$, and clearly $U_+ = U \mid \mathscr{K}_+$. It follows at once that U becomes a dilation of T upon the identification of $\mathscr{H}$ with $\{0\} \oplus \mathscr{H} \oplus \{0\} \subset \mathscr{K}$. In order to show that U is unitary it suffices to show that U and U^* are isometries. The fact that U is an isometry is equivalent to the identity

$$\|Tx + D_{T^*}e_0\|^2 + \|-T^*e_0 + D_T x\|^2 = \|e_0\|^2 + \|x\|^2, \quad e_0 \in \mathscr{D}_{T^*},\ x \in \mathscr{H}.$$

The left-hand side of this identity can be rewritten as follows:

$$\begin{aligned}&\|Tx\|^2 + \|D_{T^*}e_0\|^2 + 2\,\mathrm{Re}(Tx, D_{T^*}e_0) + \|T^*e_0\|^2 \\ &\quad + \|D_T x\|^2 - 2\,\mathrm{Re}(D_T x, T^*e_0) \\ &\quad = \|x\|^2 + \|e_0\|^2 + 2\,\mathrm{Re}[(x, T^*D_{T^*}e_0) - (x, D_T T^* e_0)],\end{aligned}$$

and the required identity follows from (1.2) applied to T^*. The minimality of U and the fact that U^* is also an isometry are left as exercises.

As noted above, the space $\mathscr{K}_+$ constructed in the proof of Theorem 1.4 can (and shall) be considered as a subspace of $\mathscr{K}$, invariant under U:

$$U_+ = U \mid \mathscr{K}_+.$$

Thus U is also a minimal unitary dilation of U_+. In fact U^* is the minimal isometric dilation of U_+^* and therefore a different proof of Theorem 1.5 would consist in showing that the minimal isometric dilation of an operator of the form U_+^* is always unitary. We chose the above proof because it is more difficult to identify the defect space of U_+^* in terms of the original operator T.

1.6. DEFINITION. A contraction $T \in \mathscr{L}(\mathscr{H})$ is said to be *completely nonunitary* if there is no invariant subspace $\mathscr{M}$ for T such that $T \mid \mathscr{M}$ is a unitary operator.

An important consequence of Theorem 1.5 is the following.

1.7. PROPOSITION. *For every contraction $T \in \mathscr{L}(\mathscr{H})$ there exist reducing subspaces $\mathscr{H}_0$, $\mathscr{H}_1$ for T such that*

(i) $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$;

(ii) $T \mid \mathscr{H}_1$ *is completely nonunitary; and*

(iii) $T \mid \mathscr{H}_0$ *is a unitary operator.*

The spaces $\mathscr{H}_0$ and $\mathscr{H}_1$ are uniquely determined by conditions (i)–(iii).

PROOF. Let $U \in \mathscr{L}(\mathscr{K})$ be a minimal unitary dilation of T. Denote by $\mathscr{K}_0$ the reducing subspace for U generated by $\mathscr{K} \ominus \mathscr{H}$ and set $\mathscr{H}_0 = \mathscr{K} \ominus \mathscr{K}_0$. Obviously $\mathscr{H}_0 \subset \mathscr{H}$ and $Tx = Ux$, $x \in \mathscr{H}_0$, because $\mathscr{H}_0$ reduces U. Thus $\mathscr{H}_0$ is reducing for T and $T \mid \mathscr{H}_0 = U \mid \mathscr{H}_0$ is unitary. We now set $\mathscr{H}_1 = \mathscr{H} \ominus \mathscr{H}_0$ and prove that $T \mid \mathscr{H}_1$ is completely nonunitary. If $\mathscr{M} \subset \mathscr{H}$ is invariant for T and $T \mid \mathscr{M}$ is unitary then the equalities

$$||h|| = ||Th|| = ||P_{\mathscr{H}} Uh||, \quad h \in \mathscr{M},$$

imply that $Th = Uh$ for $h \in \mathscr{M}$. Thus $\mathscr{M}$ is invariant for U, $U \mid \mathscr{M}$ is unitary, and hence $\mathscr{M}$ is reducing for U. Now, $\mathscr{M}$ is orthogonal onto $\mathscr{K} \ominus \mathscr{H}$ and therefore onto $\mathscr{K}_0$; we deduce that $\mathscr{M} \subset \mathscr{H}_0$. This argument shows at once that $T \mid \mathscr{H}_1$ is completely nonunitary and that the decomposition $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$ is unique with the properties (i)–(iii).

The preceding result shows that the study of general contractions can be reduced in many cases to the study of the completely nonunitary ones.

Before proving an important property of isometric and unitary dilations we study in further detail the space of the minimal isometric dilation of a contraction $T \in \mathscr{L}(\mathscr{H})$. Let us denote by $\mathscr{K}_n$, $n = 0, 1, 2, \ldots$, the subspace of $\mathscr{K}_+$ defined as

$$\mathscr{K}_n = \mathscr{H} \oplus \underbrace{\mathscr{D}_T \oplus \mathscr{D}_T \oplus \cdots \oplus \mathscr{D}_T}_{n \text{ times}} \oplus \{0\} \oplus \cdots .$$

Thus $\mathscr{K}_0 = \mathscr{H}$ and each $\mathscr{K}_n$ is invariant for U_+^*. If we set $T_n = P_{\mathscr{K}_n} U_+ \mid \mathscr{K}_n$, then T_{n+1} is a dilation of T_n for every n. The contractions T_n can be viewed differently. For an arbitrary contraction $S \in \mathscr{L}(\mathscr{H})$ we can construct a dilation $S_\sim$ of S on $\mathscr{H} \oplus \mathscr{D}_S$ defined by

$$S_\sim(x \oplus y) = Sx \oplus D_S x. \tag{1.8}$$

Clearly then $S_\sim$ is a partial isometry and

$$\mathscr{D}_{S_\sim} = \ker S_\sim = \{0\} \oplus \mathscr{D}_S.$$

Thus if we repeat this procedure, we can construct a partial isometry $S_{\sim\sim} = (S_\sim)_\sim$ which dilates $S_\sim$, acts on $\mathscr{H} \oplus \mathscr{D}_S \oplus \mathscr{D}_S$, and is defined by

$$S_{\sim\sim}(x \oplus y \oplus z) = Sx \oplus D_S x \oplus y.$$

It is clear now that the contractions T_n considered above satisfy the relations

$$T_{n+1} = (T_n)_\sim, \quad n = 0, 1, 2, \ldots,$$

up to natural unitary equivalences.

1.9. PROPOSITION. *Let $T \in \mathscr{L}(\mathscr{H})$ and $T' \in \mathscr{L}(\mathscr{H}')$ be two contractions, and let $X \in \mathscr{L}(\mathscr{H}, \mathscr{H}')$ satisfy the intertwining relation $T'X = XT$. Then there exists an operator $Y \in \mathscr{L}(\mathscr{H} \oplus \mathscr{D}_T, \mathscr{H}' \oplus \mathscr{D}_{T'})$ such that*

(i) $Y(\{0\} \oplus \mathscr{D}_T) \subset \{0\} \oplus \mathscr{D}_{T'}$;
(ii) $P_{\mathscr{H}'} Y \mid \mathscr{H} = X$;
(iii) $||Y|| = ||X||$; *and*
(iv) $T'_\sim Y = Y T_\sim$, *where $T_\sim$ and $T'_\sim$ are the dilations of T and T' described by* (1.8).

PROOF. We may assume without loss of generality that $\|X\| = 1$. Indeed, if $X = 0$ we take $Y = 0$ and if $X \neq 0$ we can replace X by $X/\|X\|$. In order to satisfy (i) and (ii), Y must have the form

$$Y(x \oplus y) = Xx \oplus (Z(x \oplus y)), \qquad x \oplus y \in \mathscr{H} \oplus \mathscr{D}_T,$$

where $Z \in \mathscr{L}(\mathscr{H} \oplus \mathscr{D}_T, \mathscr{D}_{T'})$. The condition that $\|Y\| \leq 1$ is easily seen to be equivalent to

$$\|Z(x \oplus y)\|^2 \leq \|x\|^2 - \|Xx\|^2 + \|y\|^2 = \|D_X x \oplus y\|^2$$

and therefore there must exist $C \in \mathscr{L}(\mathscr{H} \oplus \mathscr{D}_T, \mathscr{D}_{T'})$, $\|C\| \leq 1$, such that

$$Z = C(D_X \oplus I).$$

Finally, condition (iv) is easily translated into $Z(Tx \oplus D_T X) = D_{T'} Xx$, $x \in \mathscr{H}$. Thus, in order to finish the proof, it suffices to prove the existence of a contraction $C \in \mathscr{L}(\mathscr{H} \oplus \mathscr{D}_T, \mathscr{D}_{T'})$ such that

$$C(D_X Tx \oplus D_T x) = D_{T'} Xx, \quad x \in \mathscr{H}.$$

Since C can be defined to be zero on the orthocomplement of the linear manifold $\{D_X Tx \oplus D_T x : x \in \mathscr{H}\}$, we only have to prove that

$$\|D_X Tx \oplus D_T x\| \geq \|D_{T'} Xx\|, \quad x \in \mathscr{H},$$

or, equivalently,

$$\|Tx\|^2 - \|XTx\|^2 + \|x\|^2 - \|Tx\|^2 \geq \|Xx\|^2 - \|T'Xx\|^2,$$

and this inequality follows from the commutation relation $T'X = XT$ and the fact that $\|X\| = 1$. The proposition follows.

We can now prove the following general lifting theorem.

1.10. THEOREM. *Let $T \in \mathscr{L}(\mathscr{H})$, $T' \in \mathscr{L}(\mathscr{H}')$ be two contractions and $U \in \mathscr{L}(\mathscr{K})$, $U' \in \mathscr{L}(\mathscr{K}')$ be unitary or minimal isometric dilations of T and T', respectively. Then for every $X \in \mathscr{L}(\mathscr{H}, \mathscr{H}')$ satisfying $T'X = XT$, there exists $Y \in \mathscr{L}(\mathscr{K}, \mathscr{K}')$ such that $U'Y = YU$, $\|Y\| = \|X\|$, $X = P_{\mathscr{H}'} Y \mid \mathscr{H}$, $Y\mathscr{K}_+ \subset \mathscr{K}'_+$, and*

$$Y(\mathscr{K}_+ \ominus \mathscr{H}) \subset \mathscr{K}'_+ \ominus \mathscr{H}',$$

where

$$\mathscr{K}_+ = \bigvee_{n=0}^{\infty} U^n \mathscr{H}, \qquad \mathscr{K}'_+ = \bigvee_{n=0}^{\infty} U'^n \mathscr{H}'.$$

PROOF. We consider first the case when $U = U_+ \in \mathscr{L}(\mathscr{K}_+)$ [resp. $U' = U'_+ \in \mathscr{L}(\mathscr{K}'_+)$] is the minimal isometric dilation of T [resp. T']. Denote by T_n [resp. T'_n], $n \geq 0$, the compression of U_+ [resp. T'_n] to $\mathscr{H}_n = \bigvee_{j=0}^{n} U_+^j \mathscr{H}$ [resp. $\mathscr{H}'_n = \bigvee_{j=0}^{n} U_+'^j \mathscr{H}'$], and observe that by Proposition 1.9 (and the remarks preceding it) we can find bounded operators $Y_n \in \mathscr{L}(\mathscr{H}_n, \mathscr{H}'_n)$ such that $Y_0 = X$, $P_{\mathscr{H}'_n} Y_{n+1} \mid \mathscr{H}_n = Y_n$, $Y_{n+1}(\mathscr{H}_{n+1} \ominus \mathscr{H}_n) \subset \mathscr{H}'_{n+1} \ominus \mathscr{H}'_n$, $\|Y_n\| = X$, and $T'_n Y_n = Y_n T_n$ for $n \geq 0$. It obviously follows that there exists an operator

$Y_+ \in \mathcal{L}(\mathcal{K}_+, \mathcal{K}'_+)$ such that $||Y_+|| = ||X||$, $Y_+(\mathcal{K}_+ \ominus \mathcal{H}) \subset \mathcal{K}'_+ \ominus \mathcal{H}'$, and $P_{\mathcal{K}'_n} Y_+ \mathcal{K}_n = Y_n$, $n \geq 0$ (set, e.g., $Y_+ x = \lim_{n\to\infty} Y_n P_{\mathcal{K}_n} x$, $x \in \mathcal{K}_+$). Since we also have $U_+ x = \lim_{n\to\infty} T_n P_{\mathcal{K}_n} x$, $x \in \mathcal{K}_+$, it follows that $U'_+ Y_+ = Y_+ U_+$ and the theorem follows in this case.

Assume now that U [resp. U'] is a minimal unitary dilation of T [resp. T']. Then $U_+ = U \mid \mathcal{K}_+$ [resp. $U'_+ = U' \mid \mathcal{K}'_+$] is a minimal isometric dilation of T [resp. T'] so, by what has just been proved, there exists $Y_+ \in \mathcal{L}(\mathcal{K}_+, \mathcal{K}'_+)$ such that $U'_+ Y_+ = Y_+ U_+$, $||Y_+|| = ||X||$, $P_{\mathcal{H}'} Y_+ \mid \mathcal{H} = X$, and $Y_+(\mathcal{K}_+ \ominus \mathcal{H}) \subset \mathcal{K}'_+ \ominus \mathcal{H}'$. Then U^* [resp. U'^*] is a minimal isometric dilation of U_+^* [resp. U'^*_+] and $U_+^* Y_+^* = Y_+^* U'^*_+$. By the first part of the proof (applied to Y_+^*) there exists an operator $Y \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ such that $||Y|| = ||Y_+||$, $Y^*(\mathcal{K}' \ominus \mathcal{K}'_+) \subset \mathcal{K} \ominus \mathcal{K}_+$, and $P_{\mathcal{K}_+} Y^* \mid \mathcal{K}'_+ = Y_+^*$. It follows then that $Y_+ = Y \mid \mathcal{K}_+$ and Y satisfies all the conditions of the theorem.

Finally, if U and U' are arbitrary unitary dilations, then we can obviously write $U = U_0 \oplus U_1$ [resp. $U' = U'_0 \oplus U'_1$] where U_0 [resp. U'_0] is a minimal unitary dilation of T [resp. T'] acting on $\mathcal{K}_0$ [resp. $\mathcal{K}'_0$]. If $Y_0 \in \mathcal{L}(\mathcal{K}_0, \mathcal{K}'_0)$ is such that $||Y_0|| = ||X||$, $U'_0 Y_0 = Y_0 U_0$, $P_{\mathcal{H}'} Y_0 \mid \mathcal{H} = x$, and $Y_0(\mathcal{K}_+ \ominus \mathcal{H}) \subset \mathcal{K}'_+ \ominus \mathcal{H}'$, then $Y = Y_0 \oplus 0_{\mathcal{K} \ominus \mathcal{K}_0}$ will satisfy the conditions of the theorem. The proof is now complete.

A consequence of the preceding result is the following commutant lifting theorem, whose proof is left as an exercise.

1.11. THEOREM. *Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction and $U \in \mathcal{L}(\mathcal{K})$ a unitary or isometric dilation of T. For every $X \in \{T\}'$ there exists $Y \in \{U\}'$ such that*

$$Y\mathcal{K}_+ \subset \mathcal{K}_+, \qquad Y(\mathcal{K}_+ \ominus \mathcal{H}) \subset \mathcal{K}_+ \ominus \mathcal{H},$$

and $P_{\mathcal{H}} Y \mid \mathcal{H} = X$, where $\mathcal{K}_+ = \bigvee_{n=0}^{\infty} U^n \mathcal{H}$.

This commutant lifting theorem gives a very useful description of $\{T\}'$, especially so when $\{U\}'$ is easy to describe.

Exercises

1. Let S be a multiplicative semigroup of operators on $\mathcal{K}$ and assume that $\mathcal{H} \subset \mathcal{K}$ is a subspace with the property that the mapping $A \mapsto P_{\mathcal{H}} A \mid \mathcal{H}$ is multiplicative on S. Show that $\mathcal{H}$ is *semi-invariant* for S, i.e., there exist subspaces $\mathcal{M}$ and $\mathcal{N}$, invariant under every operator in S, such that $\mathcal{M} \supset \mathcal{N}$ and $\mathcal{H} = \mathcal{M} \ominus \mathcal{N}$.

2. Complete the proof of Lemma 1.3.

3. Let $U \in \mathcal{L}(\mathcal{K})$ be a minimal unitary dilation of $T \in \mathcal{L}(\mathcal{H})$ and set $U_+ = U \mid \mathcal{K}_+$, where $\mathcal{K}_+ = \bigvee_{n=0}^{\infty} U^n \mathcal{H}$. Show that U is a minimal isometric dilation of U_+^*.

4. Let $V \in \mathcal{L}(\mathcal{H})$ be an isometry. Show directly that the minimal isometric dilation of V^* is a unitary operator.

5. Let $T \in \mathscr{L}(\mathscr{H})$ be a contraction and let $\mathscr{M}$ be a subspace of $\mathscr{H}$ such that $P_{\mathscr{M}}T \mid \mathscr{M}$ is a unitary operator. Prove that $\mathscr{M}$ is a reducing subspace for T.

6. Let $T \in \mathscr{L}(\mathscr{H})$ be a contraction. Show that the matrix $U = \begin{bmatrix} T & D_T \\ -D_{T^*} & T^* \end{bmatrix}$ defines a unitary operator on $\mathscr{H} \oplus \mathscr{H}$. Is U is a unitary dilation of T?

7. Under what conditions is the operator Y in Proposition 1.9 uniquely determined?

8. Describe all operators Y satisfying conditions (i), (ii), and (iv) of Proposition 1.9.

9. Prove that if X is unitary in Proposition 1.9, then Y is uniquely determined and unitary.

10. Prove that if X is an isometry in Proposition 1.9, then Y can be chosen to be an isometry. Is Y necessarily unique?

11. Is Theorem 1.10 true if U and U' are only assumed to be isometric dilations of T and T', respectively?

12. Prove Theorem 1.11.

13. Let $T, T' \in \mathscr{L}(\mathscr{H})$ be two commuting contractions ($TT' = T'T$). Prove that there exists a Hilbert space $\mathscr{K} \supset \mathscr{H}$ and commuting unitary operators U, $U' \in \mathscr{L}(\mathscr{K})$ such that U and U' are unitary dilations of T and T', respectively.

14. If X in Theorem 1.11 belongs to the double commutant $\{T\}''$, can Y always be chosen in $\{U\}''$?

2. Isometries and unitary operators. The results of the preceding paragraph show the importance of understanding the structure and relative position of the invariant subspaces of an isometry. Let us first recall that an isometry $V \in \mathscr{L}(\mathscr{H})$ is a *unilateral shift* if there is a closed subspace $\mathscr{F} \subset \mathscr{H}$ (called a *wandering space*) such that the spaces $\{V^n\mathscr{F}\}_{n=0}^{\infty}$ are mutually orthogonal and

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} V^n \mathscr{F}.$$

The dimension of the Hilbert space $\mathscr{F}$ is called the *multiplicity* of V. Clearly a unilateral shift is a completely nonunitary operator; indeed $\bigcap_{n=0}^{\infty} V^n\mathscr{H} = \{0\}$. The following result of Wold and von Neumann shows that the converse is also true.

2.1. THEOREM. *Let V be an isometry on the Hilbert space $\mathscr{H}$. Then there exists a unique reducing subspace $\mathscr{H}_0$ for V such that*

(i) *$V \mid \mathscr{H}_0$ is a unilateral shift; and*
(ii) *$V \mid \mathscr{H} \ominus \mathscr{H}_0$ is a unitary operator.*

PROOF. The sequence of subspaces $\{V^n\mathscr{H}\}_{n=0}^{\infty}$ is obviously decreasing so that we have

$$\mathscr{H} = \left(\bigoplus_{n=0}^{\infty}(V^n\mathscr{H} \ominus V^{n+1}\mathscr{H})\right) \oplus \left(\bigcap_{n=0}^{\infty} V^n\mathscr{H}\right); \tag{2.2}$$

we set $\mathscr{H}_0 = \bigoplus_{n=0}^{\infty}(V^n\mathscr{H} \ominus V^{n+1}\mathscr{H}) = \bigoplus_{n=0}^{\infty} V^n\mathscr{F}$, $\mathscr{F} = \mathscr{H} \ominus V\mathscr{H}$. Thus $\mathscr{F}$ is a wandering subspace and $V \mid \mathscr{H}_0$ is a shift. Relation (2.2) shows that $\mathscr{H}_0$ is reducing and

$$\mathscr{H} \ominus \mathscr{H}_0 = \bigcap_{n=0}^{\infty} V^n\mathscr{H}.$$

Thus $V(\mathscr{H} \ominus \mathscr{H}_0) = \bigcap_{n=1}^{\infty} V^n\mathscr{H} = \bigcap_{n=0}^{\infty} V^n\mathscr{H} = \mathscr{H} \ominus \mathscr{H}_0$ and $\mathscr{H}_0$ has properties (i) and (ii) of the statement. Now $V \mid \mathscr{H}_0$ is completely nonunitary so that the uniqueness of $\mathscr{H}_0$ follows from Proposition 1.7.

2.3. REMARK. The projection onto $V^n\mathscr{H}$ is given by $V^nV^{*^n}$ and therefore the projection $P_{\mathscr{H}\ominus\mathscr{H}_0}$ equals the strong limit of the decreasing sequence $\{V^nV^{*^n} : n \geq 0\}$.

2.4. COROLLARY. *An isometry $V \in \mathscr{L}(\mathscr{H})$ is a unilateral shift if and only if $\lim_{n\to\infty} \|V^{*^n}x\| = 0$ for all $x \in \mathscr{H}$.*

PROOF. Assume V is a shift so that

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} V^n\mathscr{F}, \quad \mathscr{F} \subset \mathscr{H}.$$

Then $V^{*^{n+1}}x = 0$ for $x \in V^n\mathscr{F}$. Since the sequence $\{V^{*^n}\}_{n=1}^{\infty}$ is bounded in norm and the spaces $\{V^n\mathscr{F}\}_{n=0}^{\infty}$ span $\mathscr{H}$, it follows that $\lim_{n\to\infty} \|V^{*^n}x\| = 0$ for every $x \in \mathscr{H}$. Conversely, if V is not a shift and $\mathscr{H}_0$ is as in Theorem 2.1, then $\|V^{*^n}x\| = \|x\|$ for every $x \in \mathscr{H} \ominus \mathscr{H}_0$. The corollary follows.

2.5. DEFINITION. Let $T \in \mathscr{L}(\mathscr{H})$ be a contraction and $U_+ \in \mathscr{L}(\mathscr{K}_+)$ be the minimal isometric dilation of T. Set $\mathscr{L}_* = \mathscr{K}_+ \ominus U_+\mathscr{K}_+$,

$$\mathscr{M} = \bigoplus_{n=0}^{\infty} U_+^n\mathscr{L}_*, \qquad \mathscr{R} = \mathscr{K}_+ \ominus \mathscr{M}, \qquad R = U_+ \mid \mathscr{R}.$$

The space $\mathscr{R}$ is called the *residual part* of $\mathscr{K}_+$ and the unitary operator R is called the *residual part* of U_+.

There is one important particular case in which the residual part of $\mathscr{K}_+$ is absent; the part $\mathscr{M}$ is absent only when T is a unitary operator, as we shall see shortly.

2.6. DEFINITION. A contraction $T \in \mathscr{L}(\mathscr{H})$ is said to be of class $C_{\cdot 0}$ if $\lim_{n\to\infty} \|T^{*^n}x\| = 0$ for all $x \in \mathscr{H}$; T is of class $C_{0\cdot}$ if T^* is of class $C_{\cdot 0}$. Finally, T is of class C_{00} if it is both of class $C_{\cdot 0}$ and of class $C_{0\cdot}$.

2.7. PROPOSITION. *Let $T \in \mathscr{L}(\mathscr{H})$ be a contraction with minimal isometric dilation $U_+ \in \mathscr{L}(\mathscr{K}_+)$. Then the residual part of $\mathscr{K}_+$ is $\{0\}$ if and only if T is of class $C_{\cdot 0}$.*

PROOF. Assume first that T is of class $C_{\cdot 0}$, $x \in \mathscr{H}$, and k is a natural number. If $n > k$ we have

$$U_+^{*^n} U_+^k x = U_+^{*^{n-k}} x = T^{*^{n-k}} x$$

and consequently

$$\lim_{n\to\infty} \|U_+^{*^n} U_+^k x\| = 0.$$

Since the family of vectors $\{U_+^k x : x \in \mathscr{H},\ k \geq 0\}$ spans $\mathscr{K}_+$, we conclude that U_+^* is of class $C_{\cdot 0}$ and hence it is a unilateral shift by Corollary 2.4. Thus $\mathscr{R} = \{0\}$ by the von Neumann–Wold decomposition theorem.

Conversely, we note that for $x \in \mathscr{H}$ we have by Remark 2.3

$$P_{\mathscr{R}} x = \lim_{n\to\infty} U_+^n U_+^{*^n} x = \lim_{n\to\infty} U_+^n T^{*^n} x \tag{2.8}$$

so that $\|P_{\mathscr{R}} x\| = \lim_{n\to\infty} \|T^{*^n} x\|$. If $\mathscr{R} = \{0\}$ it obviously follows that T is of class $C_{\cdot 0}$.

2.9. PROPOSITION. *With the notation of Definition 2.5, the spaces $\mathscr{L}_*$ and $\mathscr{D}_{T^*}$ have the same dimension. If U_+ is the dilation constructed in the proof of Theorem 1.4, then an isometry ϕ_* of $\mathscr{D}_{T^*}$ onto $\mathscr{L}_*$ is given by*

$$\phi_*(x) = D_{T^*} x \oplus (-T^* x) \oplus 0 \oplus 0 \oplus \cdots.$$

PROOF. We have $\mathscr{L}_* = \ker U_+^*$. Assume U_+ is constructed as in Theorem 1.4. Then every element u in $\mathscr{K}_+$ can be written as

$$u = y \oplus \left(\bigoplus_{n=0}^{\infty} d_n \right)$$

with $y \in \mathscr{H}$, $d_n \in \mathscr{D}_T$, and $\sum_{n=0}^{\infty} \|d_n\|^2 < \infty$. Then we have

$$U_+^* u = (T^* y + D_T d_0) \oplus \left(\bigoplus_{n=0}^{\infty} d_{n+1} \right)$$

so that $U_+^* u = 0$ if and only if

$$T^* y + D_T d_0 = 0 \tag{2.10}$$

and $d_n = 0$ for $n \geq 1$. Upon multiplication by T, (2.10) becomes

$$TT^* y + TD_T d_0 = 0$$

or

$$y - D_{T^*}^2 y + D_{T^*} T d_0 = 0,$$

where we have used (1.1) and (1.2). Thus we have $y = D_{T^*} x$ with $x = D_{T^*} y - Td_0$. Note that $x \in \mathscr{D}_{T^*}$ since $T\mathscr{D}_T \subset \mathscr{D}_{T^*}$. Then multiplication of (2.10) by D_T easily yields

$$T^* D_{T^*} y + d_0 - T^* T d_0 = 0$$

or, equivalently, $d_0 = -T^* x$. These calculations show that the map ϕ_* is onto $\mathscr{L}_*$. That ϕ_* is an isometry is easy to verify, thus concluding the proof.