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Extensions of Linear-Quadratic Control, Optimization and Matrix Theory

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PREFACE

Control, optimization and matrix theory are closely linked in many ways, perhaps most strongly by the linear-quadratic aspects they have in common. The present work seeks to extend, develop and strengthen this link by presenting a number of extensions of the well-known linear-quadratic theories. Consequently it should prove to be particularly useful to graduate students, teachers and researchers in science and engineering.

In a very definite sense this is a personal volume - it reflects my attempts over the past five years to understand and analyse non-linear systems and to contribute new developments. Inevitably some of the material presented has previously appeared in one or another form elsewhere in the literature but many results are being made known here for the first time.

Certain of the results presented in Chapters 2, 3, 5 and 6 were developed jointly with Drs. J.L. Speyer, W.M. Getz, M. Pachter and C.A. Botsaris; their contribution and cooperation is gratefully acknowledged. Dr. D.J. Bell kindly perused the draft manuscript.

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David H. Jacobson
Pretoria, 1977

CONTENTS

1. INTRODUCTION

The treatments of linear-quadratic control problems given in [1] are probably the most comprehensive available, though by now a little dated in some respects. Both discrete-time and continuous-time formulations are treated in that reference, and variational and dynamic programming techniques are used in the generation of solutions. A perusal of [1] thus provides a rather good sample of formulations and techniques, in addition to a good list of textbooks and other references.

In this short chapter we do not very exhaustively review the material in [1] (most of which is also available elsewhere), but rather describe loosely but adequately what, for the purpose of this monograph, constitutes a linear-quadratic formulation in control, optimization or matrix theory. We then outline the contents of the following chapters in some detail in order to elucidate the magnitude of the 'extensions' presented.

Broadly speaking a linear-quadratic (Gaussian) control (variational) formulation consists of a finite-dimensional linear, discrete- or continuous-time, dynamic system which is to be controlled in such a way as to minimize the value (or expected value) of a performance criterion which is the integral, or sum, of quadratic functions of the system state and control variables plus, perhaps, a quadratic function of the state at the terminal time. Stochastic formulations allow additive Gaussian white noise to disturb the dynamic system, and the outputs that can be measured are assumed to be linear functions of the state and Gaussian white noise. The most celebrated property of the solution of the nonsingular linear-quadratic-problem (often referred to as an LQP or LQG) is that the optimal control is a linear (time-varying) function of the state or, in the stochastic case, the best estimate of the state. The matrix Riccati equation and, in the singular case [2], matrix inequalities play a special role in ensuring the existence of the solution. Questions and assumptions relating to stability, controllability and observability of the system are also important here.

By a linear-quadratic formulation in matrix theory we mean the study of the properties of positive (semi-) definite quadratic functions of a finite number of variables and their relation to linear equalities and inequalities. Here it is simply the definiteness of the quadratic function that makes the formulation standard. If this assumption is relaxed we have immediately a non-convex quadratic function which has properties not shared by the convex (positive semi-definite) case.

We use the term 'optimization', as distinct from 'optimal control', to describe the problem of finding a minimum of a function of a finite number of variables subject to equality and inequality constraints. 'Linear-quadratic' in this context refers to the fact that algorithms for the solution of the minimization problem are almost always based upon a model based in turn on the assumptions that the function to be minimized is a positive-definite quadratic form and that the constraints are linear.

It is fairly evident from the foregoing descriptions that the linear-quadratic thread that runs through control, optimization and matrix theory forms a strong conceptual and operational tie between them. Consequently in this monograph control, optimization and matrix theory are not strictly confined to separate chapters: in fact each of them is concerned with these three subjects of study to a greater or lesser extent.

In Chapter 2 we 'extend' the linear-quadratic control problem by first replacing the quadratic performance criterion by the exponential of a quadratic function. In the deterministic case we gain nothing by this move, as minimization of an exponential of a function is equivalent to minimization of that functional, but in the stochastic case a new, interesting formulation results. If the state is perfectly measurable but Gaussian white noise enters linearly into the linear system, we find that the optimal feedback controller is linear, as in the linear-quadratic case, but that the controller depends upon the statistics of the noise, unlike that for the linear-quadratic case. It turns out, interestingly, that the controller is equivalent to that obtained when the noise is treated as a 'belligerent player' in a two-person zero-sum linear-quadratic game, and this provides new justification for this type of 'worst case' controller design. If the measurement of the state is noise-corrupted, the optimal feedback controller retains its linear character but is in the general case infinite-dimensional. This is another surprise when it is recalled that in the linear-quadratic Gaussian case the controller turns out to be the finite-dimensional optimal controller for the deterministic case simply with the state replaced by the best (Kalman) filtered estimate of the state. An application due to Speyer of the exponential formulation and solution to homing missile guidance is also mentioned.

The next 'extension' is obtained by generalizing the linear dynamic system to a class of restricted non-linear stochastic systems while retaining the quadratic performance criterion. The optimal controller is here linear in the system state but depends upon the noise parameters. Known results due to Wonham, McLane and Kleinmann for linear systems with multiplicative noise are generalized here.

Next, we turn to the class of non-linear systems homogeneous-in-the-input. We demonstrate that such systems are asymptotically stabilizable under certain conditions which are almost necessary and sufficient. Furthermore, we show that stabilizing controllers actually minimize a wide variety of non-quadratic performance criteria.

We also obtain the solution to the problem of minimizing a certain non-quadratic performance criterion subject to a linear dynamic constraint. This result is generalized by Speyer to a stochastic version which involves control of a linear stochastic dynamic system driven by additive and state-dependent white-noise processes.

Finally in this chapter we study the control of systems of quadratic and bilinear differential equations and obtain some limited results, viz. that for a certain class of problems the optimal feedback controller is linear.

Taken as a whole, Chapter 2 illustrates that the linear-quadratic control problem has been extended in non-trivial ways both by using performance criteria more general than quadratic and by introducing classes of non-linear dynamic systems. These give rise to both linear and non-linear controllers.

In Chapter 3 we begin with matrix theory. First, copositive matrices are introduced. Quite simply a symmetric matrix is copositive if its associated quadratic form is non-negative for all vectors having non-negative elements. Interestingly, it turns out that all copositive matrices are sums of positive semi-definite matrices and matrices with non-negative elements (non-negative matrices) if and only if the dimensionality of the matrix is less than five. We show that this also implies that all positive semi-definite non-negative matrices have non-negative factorizations if and only if they are of dimension less than five. We show further that the representation for copositive matrices extends beyond dimension five if a more general type of copositivity, viz. stochastic copositivity, is defined.

Closely related to copositive quadratic forms is the question of non-negativity of a quadratic form subject to equality and inequality quadratic constraints. In the case of one constraint Finsler's theorem provides a complete answer, and in the case of an arbitrary number of constraints we extend Finsler's theorem to provide a useful sufficient condition. We use this extension to yield insight into the properties of the inverse of copositive matrices.

We then turn to symmetric M-matrices which are in fact positive-definite, and whose inverses are both positive-definite and non-negative. We show that these inverses have non-negative factorizations.

Next we apply copositive matrix theory to the non-convex quadratic programming problem to provide sufficient conditions for optimality.

The remainder of Chapter 3 is concerned with a study of the behaviour of solutions of systems of autonomous quadratic differential equations. Specifically we develop two sets of sufficient conditions for a solution to exhibit a finite escape time. The first set is similar to certain conditions obtained by Freeman, while the second set, being based upon our results for non-convex quadratic programming derived earlier, is less restrictive owing to our non-trivial use of the notion of invariant sets.

Chapter 4 contains what we believe are significant extensions of our results in [2] for the non-negativity of quadratic functionals. First we review and reformulate certain important sufficient conditions for the non-negativity of unconstrained quadratic functionals and extend these to the case where the control variables are constrained. A novel Riccati differential equation results from this approach. Next we further extend these sufficient conditions to a general class of non-quadratic, non-linear, constrained problems. Our results bear a resemblance to certain controllability conditions derived by Kunzi and Davison, and allow us to relate the non-negativity of non-quadratic functionals to that of a class of non-autonomous quadratic functionals.

Chapter 5 is concerned with the controllability of autonomous linear dynamic systems in which the control variables are constrained to lie within a certain constraint set. It is well known that, provided zero belongs to the interior of the convex hull of the constraint set, such a linear system is null-controllable if and only if it is completely controllable when the constraint is removed. More recently Brammer has provided necessary and sufficient conditions for null-controllability when zero does not belong to the interior of the convex hull of the constraint set. Arbitrary-interval null-controllability, introduced in Chapter 5, requires that the system be controllable on any time interval, this being a more demanding requirement than null-controllability. As is well known, a system is arbitrary-interval null-controllable if it is null-controllable and if zero belongs to the interior of the convex hull of the constraint set. The main purpose of Chapter 5, then, is to provide necessary and sufficient conditions for arbitrary-interval null-controllability when the constraint set is of general type. Most interesting is the role of arbitrary-interval null-controllability as a necessary and sufficient condition for continuity of the minimum time function in time-optimal control of an autonomous linear dynamic system.

In Chapter 6 we proceed to function minimization. We discuss the properties of a homogeneous model in comparison with a quadratic model and refer to a convergent algorithm for use on general functions. We also refer to the recent work of Kowalik who has further improved the effectiveness of the algorithm by introducing a highly stable numerical method in place of the Householder updating used in the first versions of the homogeneous algorithms.

Next in Chapter 6 we introduce the differential descent approach presented in [3] and further developed extensively by Botsaris. In this approach curvilinear, as opposed to linear search paths are used, which are developed by approximating the trajectories of steepest descent in appropriate ways. Such methods have considerable advantages in that they do not fail when Newton's method does, and automatically behave as gradient methods when far from the minimum of the function to be minimized, and as Newton's method when in the neighbourhood of the minimum.

Chapter 7 briefly assesses the earlier ones and indicates areas for further research.

1.1 References

[1] IEEE Transactions on Automatic Control, vol. AC-19, December 1971, 'Special Issue on the Linear-Quadratic-Gaussian Problem'.

[2] BELL, D.J. & JACOBSON, D.H. Singular Optimal Control Problems. Academic Press, New York and London, 1975.

[3] BOTSARIS, C.A. Differential Descent Methods for Function Minimization. Ph.D. Thesis, University of the Witwatersrand, Johannesburg, South Africa, 1975.

2. NON-LINEAR-QUADRATIC CONTROL PROBLEMS

2.1 Exponential Performance Criterion - Perfect Measurements

We consider in this section the optimal control of a linear discrete-time dynamic system disturbed by additive Gaussian white noise. In place of a quadratic performance criterion we use an exponential one [1]-[3]. We assume that the state of the system can be measured perfectly.

The assumption of Gaussian noise is deliberate - indeed it is the exponential nature of the Gaussian density function which matches the exponential nature of the performance criterion and results in the linear form of the optimal feedback controller.

2.1.1 Discrete-time Formulation

We consider a linear discrete-time dynamic system described by

$$x_{k+1} = A_k x_k + B_k u_k + \Gamma_k \omega_k, \quad k=0,\dots,N-1; \quad x_o \text{ given} \tag{2.1.1}$$

where the 'state' vector $x_k \varepsilon R^n$, the control vector $u_k \varepsilon R^m$, and the Gaussian noise input $\omega_k \varepsilon R^q$. The known matrices A_k, B_k, Γ_k have appropriate dimensions and may vary as a function of the index k.

The noise input is a sequence of independently distributed Gaussian random variables having probability density

$$p_\omega(\omega_o,\dots,\omega_{N-1}) = \prod_{k=o}^{N-1} p(\omega_k;k) \tag{2.1.2}$$