Graduate Texts in Mathematics

Béla Bollobás

Graph Theory

An Introductory Course

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There is no permanent place in the world for ugly mathematics.

G. H. Hardy
A Mathematician's Apology

Preface

This book is intended for the young student who is interested in graph theory and wishes to study it as part of his mathematical education. Experience at Cambridge shows that none of the currently available texts meet this need. Either they are too specialized for their audience or they lack the depth and development needed to reveal the nature of the subject.

We start from the premise that graph theory is one of several courses which compete for the student's attention and should contribute to his appreciation of mathematics as a whole. Therefore, the book does not consist merely of a catalogue of results but also contains extensive descriptive passages designed to convey the flavour of the subject and to arouse the student's interest. Those theorems which are vital to the development are stated clearly, together with full and detailed proofs. The book thereby offers a leisurely introduction to graph theory which culminates in a thorough grounding in most aspects of the subject.

Each chapter contains three or four sections, exercises and bibliographical notes. Elementary exercises are marked with a $^-$ sign, while the difficult ones, marked by $^+$ signs, are often accompanied by detailed hints. In the opening sections the reader is led gently through the material: the results are rather simple and their easy proofs are presented in detail. The later sections are for those whose interest in the topic has been excited: the theorems tend to be deeper and their proofs, which may not be simple, are described more rapidly. Throughout this book the reader will discover connections with various other branches of mathematics, including optimization theory, linear algebra, group theory, projective geometry, representation theory, probability theory, analysis, knot theory and ring theory. Although most of these connections are not essential for an understanding of the book, the reader would benefit greatly from a modest acquaintance with these subjects.

The bibliographical notes are not intended to be exhaustive but rather to guide the reader to additional material.

I am grateful to Andrew Thomason for reading the manuscript carefully and making many useful suggestions. John Conway has also taught the graph theory course at Cambridge and I am particularly indebted to him for detailed advice and assistance with Chapters II and VIII. I would like to thank Springer-Verlag and especially Joyce Schanbacher for their efficiency and great skill in producing this book.

Cambridge
April 1979

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Contents

CHAPTER I

Fundamentals

The purpose of this introduction is to familiarise the reader with the basic concepts and results of graph theory. The chapter inevitably contains a large number of definitions and in order to prevent the reader growing weary we prove simple results as soon as possible. The reader is not expected to have complete mastery of Chapter I before sampling the rest of the book, indeed, he is encouraged to skip ahead since most of the terminology is self-explanatory. We should add at this stage that the terminology of graph theory is far from being standard, though that used in this book is well accepted.

§1 Definitions

A *graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set of unordered pairs of V. Unless it is explicitly stated otherwise, we consider only finite graphs, that is V and E are always finite. The set V is the set of *vertices* and E is the set of *edges*. If G is a graph then $V = V(G)$ is the vertex set of G and $E = E(G)$ is the edge set. An edge $\{x, y\}$ is said to *join* the vertices x and y and is denoted by xy. Thus xy and yx mean exactly the same edge; the vertices x and y are the *endvertices* of this edge. If $xy \in E(G)$ then x and y are *adjacent* or *neighbouring* vertices of G and the vertices x and y are *incident* with the edge xy. Two edges are *adjacent* if they have exactly one common endvertex.

As the terminology suggests, we do not usually think of a graph as an ordered pair, but as a collection of vertices some of which are joined by

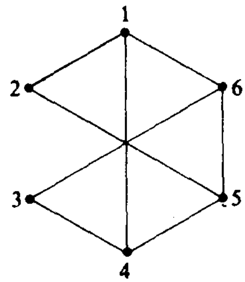

Figure I.1. A graph.

edges. It is then a natural step to draw a picture of the graph. In fact, sometimes the easiest way to describe a graph is to draw it; the graph $G = (\{1, 2, 3, 4, 5, 6\}, \{12, 14, 16, 25, 34, 36, 45, 56\})$ is immediately comprehended by looking at Figure I.1.

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. In this case we write $G' \subset G$. If G' contains *all edges* of G that join two vertices in V' then G' is said to be the subgraph *induced* or *spanned by* V' and is denoted by $G[V']$. A subgraph G' of G is an *induced* subgraph if $G' = G[V(G')]$. If $V' = V$, then G' is said to be a *spanning* subgraph of G. These concepts are illustrated in Figure I.2.

We shall often construct new graphs from old ones by deleting or adding some vertices and edges. If $W \subset V(G)$ then $G - W = G[V \backslash W]$ is the subgraph of G obtained by deleting the vertices in W *and all edges incident with them*. Similarly if $E' \subset E(G)$ then $G - E' = (V(G), E(G) \backslash E')$. If $W = \{w\}$ and $E' = \{xy\}$ then this notation is simplified to $G - w$ and $G - xy$. Similarly, if x and y are non-adjacent vertices of G then $G + xy$ is obtained from G by joining x to y.

If x is a vertex of a graph G then instead of $x \in V(G)$ we usually write $x \in G$. The *order* of G is the number of vertices; it is denoted by $|G|$. The same notation is used for the number of elements (cardinality) of a *set*: $|X|$ denotes the number of elements of the set X. Thus $|G| = |V(G)|$. The *size* of G is the number of edges; it is denoted by $e(G)$. We write G^n for an

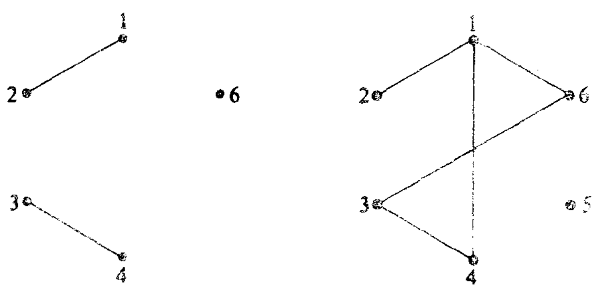

Figure I.2. A subgraph, an induced subgraph and a spanning subgraph of the graph in Figure I.1.

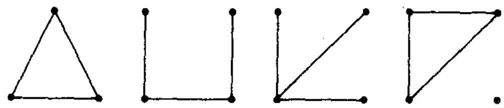

Figure I.3. Graphs of order at most 4 and size 3.

arbitrary graph of order n. Similarly $G(n, m)$ denotes an *arbitrary graph of order n and size m*.

Two graphs are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency. Thus $G = (V, E)$ is isomorphic to $G' = (V', E')$ if there is a bijection $\phi: V \to V'$ such that $xy \in E$ iff $\phi(x)\phi(y) \in E'$. Clearly isomorphic graphs have the same order and size. Usually we do not distinguish between isomorphic graphs, unless we consider graphs with a distinguished or labelled set of vertices (for example, subgraphs of a given graph). In accordance with this convention, if G and H are isomorphic graphs then we write either $G \cong H$ or simply $G = H$. In Figure I.3 we show all graphs (within isomorphism) that have order at most 4 and size 3.

The size of a graph of order n is at least 0 and at most $\binom{n}{2}$. Clearly for every m, $0 \le m \le \binom{n}{2}$, there is a graph $G(n, m)$. A graph of order n and size $\binom{n}{2}$ is called a *complete n-graph* and is denoted by K^n; an *empty n-graph* E^n has order n and no edges. In K^n every two vertices are adjacent, while in E^n no two vertices are adjacent. The graph $K^1 = E^1$ is said to be *trivial*.

The set of vertices adjacent to a vertex $x \in G$ is denoted by $\Gamma(x)$. The *degree* of x is $d(x) = |\Gamma(x)|$. If we want to emphasize that the underlying graph is G then we write $\Gamma_G(x)$ and $d_G(x)$; a similar convention will be adopted for other functions depending on an underlying graph. Thus if $x \in H = G[W]$ then

$$\Gamma_H(x) = \{y \in H : xy \in E(H)\} = \Gamma_G(x) \cap W.$$

The *minimum degree* of the vertices of a graph G is denoted by $\delta(G)$ and the *maximum degree* by $\Delta(G)$. A vertex of degree 0 is said to be an *isolated* vertex. If $\delta(G) = \Delta(G) = k$, that is every vertex of G has degree k then G is said to be *k-regular* or *regular of degree k*. A graph is *regular* if it is k-regular for some k. A 3-regular graph is said to be *cubic*.

If $V(G) = \{x_1, x_2, \ldots, x_n\}$ then $(d(x_i))_1^n$ is a *degree sequence* of G. Usually we order the vertices in such a way that the degree sequence obtained in this way is monotone increasing or monotone decreasing, for example $\delta(G) = d(x_1) \le \cdots \le d(x_n) = \Delta(G)$. Since each edge has two endvertices, the sum of the degrees is exactly twice the number of edges:

$$\sum_1^n d(x_i) = 2e(G). \tag{1}$$

In particular, the sum of degrees is even:

$$\sum_1^n d(x_i) \equiv 0 \pmod 2. \tag{2}$$

This last observation is sometimes called the *handshaking lemma*, since it expresses the fact that in any party the total number of hands shaken is even. Equivalently, (2) states that the number of vertices of *odd* degree is *even*. We see also from (1) that $\delta(G) \le \lfloor 2e(G)/n \rfloor$ and $\Delta(G) \ge \lceil 2e(G)/n \rceil$. Here $\lfloor x \rfloor$ denotes the greatest integer not greater than x and $\lceil x \rceil = -\lfloor -x \rfloor$.

A *path* is a graph P of the form

$$V(P) = \{x_0, x_1, \ldots, x_l\}, \qquad E(P) = \{x_0x_1, x_1x_2, \ldots, x_{l-1}x_l\}.$$

This path P is usually denoted by $x_0x_1 \ldots x_l$. The vertices x_0 and x_l are the *endvertices* of P and $l = e(P)$ is the *length* of P. We say that P is a path *from* x_0 *to* x_l or an x_0-x_l *path*. Of course, P is also a path from x_l to x_0 or an x_l-x_0 path. Sometimes we wish to emphasize that P is considered to go from x_0 to x_l and then call x_0 the *initial* and x_l the *terminal* vertex of P. A path with initial vertex x is an x-*path*.

The term *independent* will be used in connection with vertices, edges and paths of a graph. A set of *vertices* (*edges*) is *independent* if no two elements of it are adjacent; a set of *paths* is *independent* if for any two paths each vertex belonging to both paths is an endvertex of both. Thus $P_1, P_2, \ldots, P_k$ are independent x-y paths iff $V(P_i) \cap V(P_j) = \{x, y\}$ whenever $i \ne j$. Also, $W \subset V(G)$ consists of independent vertices iff $G[W]$ is an empty graph.

Most paths we consider are subgraphs of a given graph G. A *walk* W in G is an alternating sequence of vertices and edges, say $x_0, \alpha_1, x_1, \alpha_2, \ldots, \alpha_l, x_l$ where $\alpha_i = x_{i-1}x_i$, $0 < i \le l$. In accordance with the terminology above, W is an x_0-x_l *walk* and is denoted by $x_0x_1 \ldots x_l$; the *length* of W is l. This walk W is called a *trail* if all its edges are distinct Note that a path is a walk with distinct vertices. A trail whose endvertices coincide (a *closed* trail) is called a *circuit*. If a walk $W = x_0x_1 \ldots x_l$ is such that $l \ge 3$, $x_0 = x_l$ and the vertices x_i, $0 < i < l$, are distinct from each other and x_0 then W is said to be a *cycle*. For simplicity this cycle is denoted by $x_1x_2 \ldots x_l$. Note that the notation differs from that of a path since x_1x_l is also an edge of this cycle. Furthermore, $x_1x_2 \ldots x_l$, $x_lx_{l-1} \ldots x_1$, $x_2x_3 \ldots x_lx_1$, $x_ix_{i-1} \ldots x_1x_lx_{l-1} \ldots x_{i+1}$ all denote the same cycle.

The symbol P^l denotes an arbitrary *path of length* l and C^l denotes a *cycle of length* l. We call C^3 a triangle, C^4 a quadrilateral, C^5 a pentagon, etc. (See Figure I.4). A cycle is *even* (*odd*) if its length is even (odd).

Given vertices x, y, their *distance* $d(x, y)$ is the minimum length of an x-y path. If there is no x-y path then $d(x, y) = \infty$.

A graph is *connected* if for every pair $\{x, y\}$ of distinct vertices there is a path from x to y. Note that a connected graph of order at least 2 cannot contain an isolated vertex. A *maximal connected subgraph* is a *component*

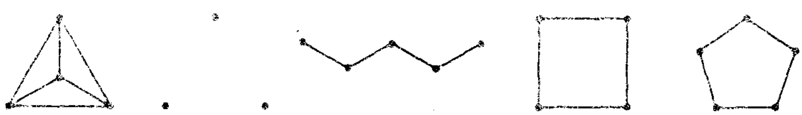

Figure I.4. The graphs K^4, E^3, P^4, C^4 and C^5.

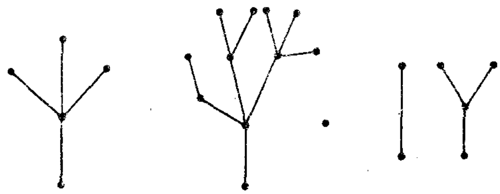

Figure I.5. A forest.

of the graph. A *cutvertex* is a vertex whose deletion increases the number of components. Similarly an edge is a *bridge* if its deletion increases the number of components. Thus an edge of a connected graph is a bridge if its deletion disconnects the graph. A graph without any cycles is a *forest* or an *acyclic* graph; a *tree* is a connected forest. (See Figure I.5.) The relation of a tree to a forest sounds less absurd if we note that a forest is a disjoint union of trees; in other words, a forest is a graph whose every component is a tree.

A graph G is a *bipartite graph with vertex classes* V_1 and V_2 if $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and each edge joins a vertex of V_1 to a vertex of V_2. Similarly G is *r-partite with vertex classes* $V_1, V_2, \ldots, V_r$ if $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$, $V_i \cap V_j = \emptyset$ whenever $1 \leq i < j \leq r$, and no edge joins two vertices in the same class. The graphs in Figure I.1 and Figure I.5 are bipartite. The symbol $K(n_1, \ldots, n_r)$ denotes a *complete r-partite graph*: it has n_i vertices in the ith class and contains all edges joining vertices in distinct classes. For simplicity we often write $K^{p,q}$ instead of $K(p, q)$ and $K_r(t)$ instead of $K(t, \ldots, t)$.

We shall write $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$ and kG for the union of k disjoint copies of G. We obtain the *join* $G + H$ from $G \cup H$ by adding all edges between G and H. Thus, for example, $K^{2,3} = E^2 + E^3$ and $K_r(t) = E^t + \cdots + E^t$.

There are several notions closely related to that of a graph. A *hypergraph* is a pair (V, E) such that $V \cap E = \emptyset$ and E is a subset of $\mathscr{P}(V)$, the power set of V, that is the set of all subsets of V. In fact, there is a simple 1-1 correspondence between the class of hypergraphs and the class of certain bipartite graphs. Indeed, given a hypergraph (V, E), construct a bipartite graph with vertex classes V and E by joining a vertex $x \in V$ to a hyperedge $S \in E$ iff $x \in S$.

By definition a graph does not contain a *loop*, an "edge" joining a vertex to itself; neither does it contain *multiple edges*, that is several "edges" joining the same two vertices. In a *multigraph* both multiple edges and multiple loops are allowed; a loop is a special edge.

If the edges are *ordered* pairs of vertices then we get the notions of a *directed graph* and *directed multigraph*. An ordered pair (a, b) is said to be an edge *directed from a to b*, or an edge *beginning at a and ending at b*, and is denoted by $\overrightarrow{ab}$ or simply ab. The notions defined for graphs are easily carried over to multigraphs, directed graphs and directed multigraphs, mutatis mutandis. Thus a (directed) trail in a directed multigraph is an