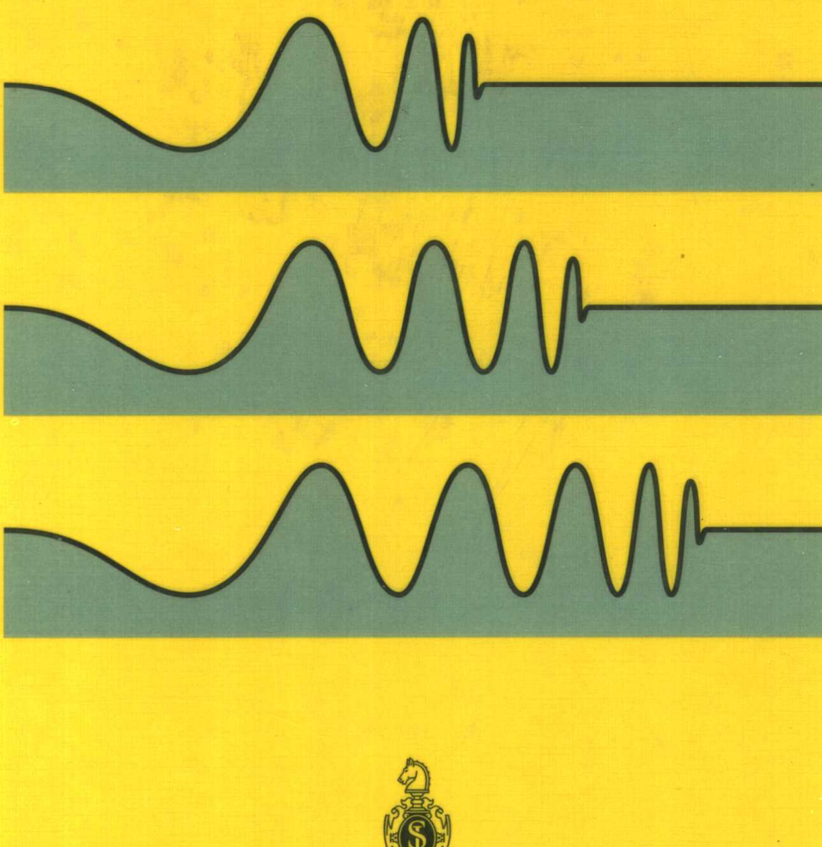
Dale R. Durran

Numerical Methods for Wave Equations in Geophysical Fluid Dynamics



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Numerical Methods for Wave Equations in Geophysical Fluid Dynamics

With 93 Illustrations

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Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM).*

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Preface

This book is designed to serve as a textbook for graduate students or advanced undergraduates studying numerical methods for the solution of partial differential equations governing wave-like flows. Although the majority of the schemes presented in this text were introduced in either the applied-mathematics or atmospheric-science literature, the focus is not on the nuts-and-bolts details of various atmospheric models but on fundamental numerical methods that have applications in a wide range of scientific and engineering disciplines. The prototype problems considered include tracer transport, shallow-water flow and the evolution of internal waves in a continuously stratified fluid.

A significant fraction of the literature on numerical methods for these problems falls into one of two categories, those books and papers that emphasize theorems and proofs, and those that emphasize numerical experimentation. Given the uncertainty associated with the messy compromises actually required to construct numerical approximations to real-world fluid-dynamics problems, it is difficult to emphasize theorems and proofs without limiting the analysis to classical numerical schemes whose practical application may be rather limited. On the other hand, if one relies primarily on numerical experimentation it is much harder to arrive at conclusions that extend beyond a specific set of test cases. In an attempt to establish a clear link between theory and practice, I have tried to follow a middle course between the theorem-and-proof formalism and the reliance on numerical experimentation. There are no formal proofs in this book, but the mathematical properties of each method are derived in a style familiar to physical scientists. At the same time, numerical examples are included that illustrate these theoretically derived properties and facilitate the intercomparison of various methods.

A general course on numerical methods for geophysical fluid dynamics might draw on portions of the material presented in Chapters 2 through 6. Chapter 2 describes the largely classical theory of finite-difference approximations to the one-way wave equation (or alternatively the constant-wind-speed advection equation). The extension of these results to systems of equations, several space dimensions, dissipative flows and nonlinear problems is discussed in Chapter 3. Chapter 4 introduces series-expansion methods with emphasis on the Fourier and spherical-harmonic spectral methods and the finite-element method. Finite-volume methods are discussed in Chapter 5 with particular attention devoted to methods for simulating the transport of scalar fields containing poorly resolved spatial gradients. Semi-Lagrangian schemes are analyzed in Chapter 6. Both theoretical and applied problems are provided at the end of each chapter. Those problems that require numerical computation are marked by an asterisk.

In addition to the core material in Chapters 2 through 6, the introduction in Chapter 1 discusses the relation between the equations governing wave-like geophysical flows and other types of partial differential equations. Chapter 1 concludes with a short overview of the strategies for numerical approximation that are considered in detail throughout the remainder of the book. Chapter 7 examines schemes for the approximation of slow moving waves in fluids that support physically insignificant fast waves. The emphasis in Chapter 7 is on atmospheric applications in which the slow wave is either an internal gravity wave and the fast waves are sound waves, or the slow wave is a Rossby wave and the fast waves are both gravity waves and sound waves. Chapter 8 examines the formulation of wave-permeable boundary conditions for limited-area models with emphasis on the shallow-water equations in one and two dimensions and on internally stratified flow.

Many numerical methods for the simulation of internally stratified flow require the repeated solution of elliptic equations for pressure or some closely related variable. Due to the limitations of my own expertise and to the availability of other excellent references I have not discussed the solution of elliptic partial differential equations in any detail. A thumbnail sketch of some solution strategies is provided in Section 7.1.3; the reader is referred to Chapter 5 of Ferziger and Perić (1997) for an excellent overview of methods for the solution of elliptic equations arising in computational fluid dynamics.

I have attempted to provide sufficient references to allow the reader to further explore the theory and applications of many of the methods discussed in the text, but the reference list is far from encyclopedic and certainly does not include every worthy paper in the atmospheric science or applied mathematics literature. References to the relevant literature in other disciplines and in foreign language journals is rather less complete.[1]

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[1] Those not familiar with the atmospheric science literature may be surprised by the number of references to *Monthly Weather Review*, which despite its title, has become the primary American journal for the publication of papers on numerical methods in atmospheric science.

This book would not have been written without the generous assistance of several colleagues. Christopher Bretherton, in particular, provided many perceptive answers to my endless questions. J. Ray Bates, Byron Boville, Michael Cullen, Marcus Grote, Robert Higdon, Randall LeVeque, Christoph Schär, William Skamarock, Piotr Smolarkiewicz, and David Williamson all provided very useful comments on individual chapters. Many students used earlier versions of this manuscript in my courses in the Atmospheric Sciences Department at the University of Washington, and their feedback helped improve the clarity of the manuscript. Two students to whom I am particularly indebted are Craig Epifanio and Donald Slinn. I am also grateful to James Holton for encouraging me to undertake this project.

It is my pleasure to acknowledge the many years of support for my numerical modeling efforts provided by the Mesoscale Dynamic Meteorology Program of the National Science Foundation. Additional support for my atmospheric simulation studies has been provided by the Coastal Meteorology ARI of the Office of Naval Research. Part of this book was completed while I was on sabbatical at the Laboratoire d'Aérologie of the Université Paul Sabatier in Toulouse, France, and I thank Daniel Guedalia and Evelyne Richard for helping make that year productive and scientifically stimulating.

As errors in the text are identified, they will be posted on the web at http://www. atmos.washington.edu/methods.for.waves, which can be accessed directly or via Springer's home page at http://www.springer-ny.com. I would be most grateful to be advised of any typographical or other errors by electronic mail at dale.durran@atmos.washington.edu.

Seattle, Washington DALE R. DURRAN

Cover art: The three curves plot solutions to the linearized Rossby–adjustment problem. The governing equations and physical parameters for this problem are identical to those given in Problem 12 of Chapter 3, except that the spatial domain is $-400 \text{ km} \leq x \leq 400 \text{ km}$ with open lateral boundaries, and the initial condition for the free-surface displacement is $h(x, t = 0) = \arctan(x/20 \text{ km})$. The curves shown are plots of $u(x, t = 943 \text{ s})$, $u(x, t = 1222 \text{ s})$, and $u(x, t = 1501 \text{ s})$ on an artisically cropped portion of the sub-domain $x > 0$.

Contents

1
Introduction

The possibility of deterministic weather prediction was suggested by Vilhelm Bjerknes as early as 1904. Around the time of the First World War, Lewis Richardson actually attempted to produce such a forecast by manually integrating a finite-difference approximation to the equations governing atmospheric motion. Unfortunately, his calculations did not yield a reasonable forecast. Moreover, the human labor required to obtain this disappointing result was so great that subsequent attempts at deterministic weather prediction had to await the introduction of a high-speed computational aid. In 1950 a team of researchers, under the direction of Jule Charney and John von Neumann at the Institute for Advanced Study, at Princeton, journeyed to the Aberdeen Proving Ground, where they worked for approximately twenty-four hours to coax a one-day weather forecast from the first general-purpose electronic computer, the ENIAC.[1] The first computer-generated weather forecast was surprisingly good, and its success led to the rapid growth of a new meteorological subdiscipline, "numerical weather prediction." These early efforts in numerical weather prediction also began a long and fruitful collaboration between numerical analysts and atmospheric scientists.[2] The use of numerical models in atmospheric and oceanic science has subsequently expanded into almost all areas of current research. Numerical models are currently employed to study phenomena as diverse as global climate change, the interaction of ocean currents with bottom topography, and the development of rotation in tornadic thunderstorms.

[1] ENIAC is an acronym for Electronic Numerical Integrator and Calculator.

[2] Further details about these early weather prediction efforts may be found in Bjerknes (1904), Richardson (1922), Charney et al. (1950), Burks and Burks (1981), and Thompson (1983).

Many of the phenomena simulated with atmospheric and oceanic models can be classified as wave-like flows if the terminology "wave-like" is used in the general sense suggested by Whitham (1974), who defined a wave as "any recognizable signal that is transferred from one part of a medium to another with a recognizable velocity of propagation." The purpose of this book is to present the fundamental mathematical aspects of a wide variety of numerical methods for the simulation of wave-like flow. The methods to be considered are typically those that have seen some use in real-world atmospheric or ocean models, but the focus is on the essential properties of each method and not on the details of any specific model. The fundamental character of each scheme will be examined in standard fluid-dynamical problems like tracer transport, shallow-water waves, and waves in an internally stratified fluid. These are the same prototypical problems familiar to many applied mathematicians, fluid dynamicists, and practitioners in the larger discipline of computational fluid dynamics.

Most of the problems under investigation in the atmospheric and oceanic sciences involve fluid systems with low viscosity and weak dissipation. The equations governing these flows are often nonlinear, but their solutions almost never develop energetic shocks or discontinuities. Nevertheless, regions of scale collapse do frequently occur as the velocity field stretches and deforms an initially compact fluid parcel. The numerical methods that will be examined in this book may therefore be distinguished from the larger family of algorithms in computational fluid mechanics in that they are particularly appropriate for low-viscosity flows, but are not primarily concerned with the treatment of shocks.

It is assumed that the reader has already been exposed to the derivation of the equations describing fluid flow and tracer transport. These derivations are given in a general fluid-dynamical context in Batchelor (1967), Yih (1977), and Bird et al. (1960), and in the context of atmospheric and oceanic science in Gill (1982), Holton (1992), and Pedlosky (1987). The mathematical properties of these equations and commonly used simplifications, such as the Boussinesq approximation, will be briefly reviewed in this chapter. The chapter concludes with a brief overview of the numerical methods that will be considered in more detail throughout the remainder of the book.

1.1 Partial Differential Equations—Some Basics

Different types of partial differential equations require different solution strategies. It is therefore helpful to begin by reviewing some of the terminology used to describe various types of partial differential equations. The *order* of a partial differential equation is the order of the highest-order partial derivative that appears in the equation. Numerical methods for the solution of time-dependent problems are often designed to solve systems of partial differential equations in which the time derivatives are of first order. These numerical methods can be used to solve partial differential equations containing higher-order time derivatives by defining new unknown functions equal to the lower-order time derivatives of the original unknown function and expressing the result as system of partial differential equations in which all time derivatives are of order one. For example, the second-order partial differential equation

$$\frac{\partial^2 \psi}{\partial t^2} + \psi \frac{\partial \psi}{\partial x} = 0$$

can be expressed as the first-order system

$$\frac{\partial v}{\partial t} + \psi \frac{\partial \psi}{\partial x} = 0,$$

$$\frac{\partial \psi}{\partial t} - v = 0.$$

In geophysical applications it is seldom necessary to actually formulate first-order-in-time equations using this procedure, because suitable first-order-in-time systems can usually be derived from fundamental physical principles.

The accurate numerical solution of equations describing wave-like flow becomes more difficult if the solution develops significant perturbations on spatial scales close to the shortest scale that can be resolved by the numerical model. The possibility of waves developing small-scale perturbations from smooth initial data increases as the governing partial differential equation becomes more nonlinear. A partial differential equation is *linear* if it is linear in the unknown functions and their derivatives, in which case the coefficients multiplying each function or derivative depend only on the independent variables. As an example,

$$\frac{\partial u}{\partial t} + x^3 \frac{\partial u}{\partial x} = 0$$

is a linear first-order partial differential equation, whereas

$$\left(\frac{\partial u}{\partial t}\right)^2 + \sin\left(u \frac{\partial u}{\partial x}\right) = 0$$

is a nonlinear first-order partial differential equation.

Analysis techniques and solution procedures developed for linear partial differential equations can be generalized most easily to the subset of nonlinear partial differential equations that are quasi-linear. A partial differential equation of order p is *quasi-linear* if it is linear in the derivatives of order p; the coefficient multiplying each pth derivative can depend on the independent variables and all derivatives of the unknown function through order $p - 1$. Two examples of quasi-linear partial differential equations are

$$\frac{\partial u}{\partial t} + u^3 \frac{\partial u}{\partial x} = 0$$

and the vorticity equation for two-dimensional nondivergent flow

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \nabla^2 \psi}{\partial x} = 0,$$

where $\psi(x, y, t)$ is the stream function for the nondivergent velocity field and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

1.1.1 *First-Order Hyperbolic Equations*

Many waves can be mathematically described as solutions to hyperbolic partial differential equations. One simple example of a hyperbolic partial differential equation is the general first-order quasi-linear equation

$$A(x, t, u)\frac{\partial u}{\partial t} + B(x, t, u)\frac{\partial u}{\partial x} = C(x, t, u), \tag{1.1}$$

where A, B, and C are real-valued functions with continuous first derivatives. This equation is hyperbolic because there exists a family of real-valued curves in the x-t plane along which the solution can be locally determined by integrating ordinary differential equations. These curves, called *characteristics*, may be defined with respect to the parameter s by the relations

$$\frac{dt}{ds} = A, \qquad \frac{dx}{ds} = B. \tag{1.2}$$

The identity

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds}$$

can then be used to express (1.1) as the ordinary differential equation

$$\frac{du}{ds} = C. \tag{1.3}$$

Given the value of u at some arbitrary point (x_0, t_0), the coordinates of the characteristic curve passing through (x_0, t_0) can be determined by integrating the ordinary differential equations (1.2). The solution along this characteristic can be obtained by integrating the ordinary differential equation (1.3). A unique solution to (1.1) can be determined throughout some local region of the x-t plane by specifying data for u along any noncharacteristic line.

In physical applications where the independent variable t represents time, the particular solution of (1.1) is generally determined by specifying initial data for u along the line $t = 0$. In such applications A is nonzero, and any perturbation in the distribution of u at the point (x_0, t_0) translates through a neighborhood of x_0 at the speed

$$\frac{dx}{dt} = \frac{B}{A}.$$

The solutions to (1.1) are wave-like in the general sense that the perturbations in u travel at well-defined velocities even though they may distort as they propagate.

The evolution of the solution is particularly simple when $C = 0$ and B/A is some constant value c, in which case (1.1) reduces to

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \tag{1.4}$$

If $u(x,0) = f(x)$, the solution to the preceding is $f(x - ct)$, implying that the initial perturbations in u translate without distortion at a uniform velocity c. Equation (1.4), which is often referred to as the *one-way wave equation* or the *constant-wind-speed advection equation*, is the simplest mathematical model for wave propagation. Although it is quite simple, (1.4) is a very useful prototype problem for testing numerical methods because solutions to more complex linear hyperbolic systems can often be expressed as the superposition of individual waves governed by one-way wave equations.

A system of partial differential equations in two independent variables is hyperbolic if it has a complete set of characteristic curves that can in principle be used to locally determine the solution from appropriately prescribed initial data. As a first example, consider a constant-coefficient linear system of the form

$$\frac{\partial u_r}{\partial t} + \sum_{s=1}^{n} a_{rs}\frac{\partial u_s}{\partial x} = 0, \qquad r = 1, 2, \ldots, n. \tag{1.5}$$

This system may be alternatively written as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0},$$

where uppercase boldface letters represent matrices and lowercase boldface letters denote vectors. The system is hyperbolic if there exist bounded matrices $\mathbf{T}$ and $\mathbf{T}^{-1}$ such that $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix with real eigenvalues d_{jj}. When the system is hyperbolic, it can be transformed to

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{D}\frac{\partial \mathbf{v}}{\partial x} = \mathbf{0} \tag{1.6}$$

by defining $\mathbf{v} = \mathbf{T}^{-1}\mathbf{u}$. Since $\mathbf{D}$ is a diagonal matrix, each element v_j of the vector of unknown functions may be determined by solving a simpler scalar equation of the form (1.4). Each diagonal element of $\mathbf{D}$ is associated with a family of characteristic curves along which the perturbations in v_i propagate at speed $dx/dt = d_{jj}$. The wave-like character of the solution can be demonstrated by Fourier transforming (1.6) with respect to x to obtain

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} + ik\mathbf{D}\hat{\mathbf{v}} = \mathbf{0}, \tag{1.7}$$

where $\hat{\mathbf{v}}$ is the Fourier transform of $\mathbf{v}$ and k is the wave number, or dual variable. In order to satisfy (1.7), the jth component of $\mathbf{v}$ must be a wave of the form