DUAL ALGEBRAS WITH APPLICATIONS TO INVARIANT SUBSPACES AND DILATION THEORY

Hari Bercovici, Ciprian Foiaş & Carl Pearcy

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This book is dedicated to Paul R. Halmos and Béla Sz.-Nagy,
who have helped and inspired the authors in many ways
over a period of many years.

Preface

This book is a slightly expanded and revised version of the lecture notes from the NSF/CBMS Regional Conference held in Tempe, Arizona in May, 1984, at which the third author was the principal lecturer. In the book we have tried to summarize some of the voluminous progress that has been made in the theory of dual algebras since the appearance in 1978 of Scott Brown's pioneering paper, which clearly showed the utility of this concept for studying the structure theory of bounded linear operators on Hilbert space. The aim of the book is to present an approach for studying (nonselfadjoint) dual algebras that allows one to obtain, in particular, results on invariant subspaces and dilation theory.

The book is put together as follows. Chapter I consists of preliminaries of a general nature concerning dual algebras. Most of Chapters II, III, IV, and X are taken from **[6a]**, but Chapters III and VI contain some new material, especially Theorem 6.3 and the results leading up to it. Chapters IV and V are taken from **[10]**, and Chapter VIII comes from **[7a]** and **[6b]**. Chapter VIII is a rewrite of part of **[11]** and **[12a]**, and Chapter IX is taken from a version of **[11a]**.

We wish to acknowledge our indebtedness to Constantin Apostol and Béla Sz.-Nagy, with whom we obtained many of the results to be found herein, and to Frank Gilfeather and the other members of the Mathematical Sciences Section of the National Science Foundation, whose support and encouragement have contributed greatly to our efforts.

Hari Bercovici
Ciprian Foiaş
Carl Pearcy

Ann Arbor, Michigan
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Tempe, Arizona
May 15, 1984

Carl Pearcy

Contents

I. Dual Algebras

In these lectures we will be interested in bounded linear operators acting on a separable, complex Hilbert space. Mostly, we will be interested in the case in which the Hilbert space is infinite dimensional, but in several topics to be treated, the finite-dimensional case is interesting and sometimes even important. Thus, *throughout the book* $\mathscr{X}$ will denote a separable complex Hilbert space whose dimension is *less than or equal to* $\aleph_0$, and $\mathscr{L}(\mathscr{X})$ will denote the algebra of all bounded linear operators on $\mathscr{X}$. Furthermore, *throughout the book* $\mathscr{H}$ will denote a separable, complex Hilbert space whose dimension is *equal to* $\aleph_0$.

With $\mathscr{X}$ as above, let $\mathbb{K} = \mathbb{K}(\mathscr{X})$ denote the norm-closed ideal of *compact* operators in $\mathscr{L}(\mathscr{X})$, and recall that $\mathbb{K}$ contains all other proper ideals in $\mathscr{L}(\mathscr{X})$. The quotient space $\mathscr{L}(\mathscr{X})/\mathbb{K}$ is a C^*-algebra which is trivial when $\dim \mathscr{X} < \aleph_0$ and is called the *Calkin algebra* when $\mathscr{X} = \mathscr{H}$. The projection of $\mathscr{L}(\mathscr{H})$ onto the Calkin algebra is denoted by π. If $K \in \mathbb{K}(\mathscr{H})$, we write the polar decomposition $K = UP$, where $P = (K^*K)^{1/2}$. Then, of course, $P \in \mathbb{K}(\mathscr{H})$ and thus has a diagonal matrix $\mathrm{Diag}(\lambda_1, \lambda_2, \ldots)$ relative to some orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathscr{H}$, where $\lambda_n \geqslant 0$ for $n \in \mathbb{N}$. This correspondence $P \sim \mathrm{Diag}(\lambda_1, \lambda_2, \ldots)$ can be used to define the Schatten p-ideals as follows.

For $p \geqslant 1$, define $\mathscr{C}_p = \mathscr{C}_p(\mathscr{H})$ to be the set of all $K = UP$ belonging to $\mathbb{K}(\mathscr{H})$ such that $\sum_{n=1}^{\infty} \lambda_n^p < +\infty$.

PROPOSITION 1.1. *For* $p \geqslant 1$, $\mathscr{C}_p(\mathscr{H})$ *is a (two-sided) ideal in* $\mathscr{L}(\mathscr{H})$ *which is a Banach *-algebra under the norm* $\|K\|_p = \{\sum_{n=1}^{\infty} \lambda_n^p\}^{1/p}$. *The family* $\{\mathscr{C}_p(\mathscr{H})\}_{p \geqslant 1}$ *is increasing, and if* $1 \leqslant p \leqslant q$, *then* $\|K\| \leqslant \|K\|_q \leqslant \|K\|_p$ *for all* K *in* $\mathscr{C}_p(\mathscr{H})$.

We will mostly be interested in $\mathscr{C}_1(\mathscr{H})$, the *trace-class*, and, to a lesser extent, $\mathscr{C}_2(\mathscr{H})$, the *Hilbert-Schmidt class*.

DEFINITION 1.2. If $\mathscr{X} \neq (0)$ is a finite-dimensional Hilbert space, then the only nonzero ideal in $\mathscr{L}(\mathscr{X})$ is $\mathscr{L}(\mathscr{X})$ itself. Nevertheless, for $p \geqslant 1$, we will write $\mathscr{C}_p(\mathscr{X})$ for $\mathscr{L}(\mathscr{X})$ equipped with the norm $\| \;\|_p$ defined similarly to what was done above, and we call $\mathscr{C}_1(\mathscr{X})$ and $\mathscr{C}_2(\mathscr{X})$ the *trace-class* and *Hilbert-Schmidt class* on $\mathscr{X}$, respectively, equipped with the *trace norm* $\| \;\|_1$ and *Hilbert-Schmidt norm* $\| \;\|_2$.

The following proposition shows where the trace-class gets its name.

PROPOSITION 1.3. *With $\mathscr{H}$ as always, there is a continuous linear functional* tr: $\mathscr{C}_1(\mathscr{H}) \to \mathbb{C}$ *on the trace-class with the property that if* $\{e_m\}_{m \in M}$ *is any orthonormal basis for* $\mathscr{H}$, *then* $\operatorname{tr}(K) = \sum_{m \in M}(Ke_m, e_m)$ *and*

$$|\operatorname{tr}(K)| \leq \|K\|_1 = \operatorname{tr}\big((K^*K)^{1/2}\big)$$

for every K *in* $\mathscr{C}_1(\mathscr{H})$. *Finally, if* $T \in \mathscr{L}(\mathscr{H})$ *and* $K \in \mathscr{C}_1(\mathscr{H})$, *then* $\operatorname{tr}(TK) = \operatorname{tr}(KT)$.

In these lectures operators of rank one will play a distinguished role, so it is worthwhile to review some elementary facts about them. If $x, y \in \mathscr{H}$, we denote by $x \otimes y$ the rank-one operator defined by $(x \otimes y)(u) = (u, y)x$ for every u in $\mathscr{H}$. One easily checks that if $A \in \mathscr{L}(\mathscr{H})$, then

$$A(x \otimes y) = (Ax) \otimes y \quad \text{and} \quad (x \otimes y)(A) = x \otimes (A^*y).$$

PROPOSITION 1.4. *For all* x, y *in* $\mathscr{H}$, $x \otimes y \in \mathscr{C}_1$, $\operatorname{tr}(x \otimes y) = (x, y)$, *and* $\|x \otimes y\|_1 = \|x \otimes y\| = \|x\|\|y\|$.

We now recall some standard duality results (cf. [**18**, p. 40]).

PROPOSITION 1.5. *With $\mathscr{H}$ as always, the dual space* $\mathscr{C}_1(\mathscr{H})^*$ *of the Banach space* $\mathscr{C}_1(\mathscr{H})$ *can be identified with* $\mathscr{L}(\mathscr{H})$. *This duality is implemented by the bilinear functional*

$$\langle T, K\rangle = \operatorname{tr}(TK), \qquad T \in \mathscr{L}(\mathscr{H}),\ K \in \mathscr{C}_1.$$

In particular, we have that

$$\|T\| = \sup\{|\langle T, K\rangle|: K \in \mathscr{C}_1, \|K\|_1 \leq 1\}$$

and

$$\|K\|_1 = \sup\{|\langle T, K\rangle|: T \in \mathscr{L}(\mathscr{H}), \|T\| \leq 1\}.$$

Throughout the remainder of these lectures we will identify $\mathscr{C}_1(\mathscr{H})^*$ with $\mathscr{L}(\mathscr{H})$ without further comment. In particular, this duality gives to $\mathscr{L}(\mathscr{H})$ a *weak* topology*, which is characterized by the fact that a net $\{T_\lambda\}$ in $\mathscr{L}(\mathscr{H})$ is weak* convergent to an operator T_0 if and only if for every $K \in \mathscr{C}_1$, $\operatorname{tr}(T_\lambda K) \to \operatorname{tr}(T_0 K)$. We now briefly review some other important topologies on $\mathscr{L}(\mathscr{H})$.

The *ultraweak operator topology* on $\mathscr{L}(\mathscr{H})$ is that locally convex, Hausdorff topology determined by saying that a net $\{T_\lambda\}$ in $\mathscr{L}(\mathscr{H})$ converges ultraweakly to T_0 if and only if, for every pair $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ of sequences from $\mathscr{H}$ such that $\sum_{n=1}^\infty\|x_n\|^2 < \infty$ and $\sum_{n=1}^\infty\|y_n\|^2 < \infty$, we have $\sum_{n=1}^\infty(T_\lambda x_n, y_n) \to \sum_{n=1}^\infty(T_0 x_n, y_n)$.

The *weak operator topology* (WOT) on $\mathscr{L}(\mathscr{H})$ is that locally convex, Hausdorff topology determined by saying that a net $\{T_\lambda\}$ in $\mathscr{L}(\mathscr{H})$ WOT-converges to T_0 if and only if, for all $x, y \in \mathscr{H}$, $(T_\lambda x, y) \to (T_0 x, y)$, or, equivalently,

$$\operatorname{tr}(T_\lambda(x \otimes y)) \to \operatorname{tr}(T_0(x \otimes y)).$$

The *strong operator topology* (SOT) on $\mathcal{L}(\mathcal{X})$ is that locally convex, Hausdorff topology determined by saying that a net $\{T_\lambda\}$ in $\mathcal{L}(\mathcal{X})$ converges strongly to T_0 if and only if, for all $x \in \mathcal{X}$, $\|(T_\lambda - T_0)x\| \to 0$.

The following proposition gives some relationships between these topologies. For more detail, see, for example, [**18**, p. 32].

PROPOSITION 1.6. *The weak* topology and the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$ coincide. This topology, which we usually call the weak* topology, and the strong operator topology are both stronger than the weak operator topology on $\mathcal{L}(\mathcal{H})$ and weaker than the norm topology on $\mathcal{L}(\mathcal{H})$. The weak* topology and strong operator topology are not comparable on $\mathcal{L}(\mathcal{H})$. If $\mathcal{B} \subset \mathcal{L}(\mathcal{H})$ is any (norm) bounded set, then the relative topologies induced on $\mathcal{B}$ by the weak* and weak operator topologies coincide. Hence the unit ball in $\mathcal{L}(\mathcal{H})$ is compact and metrizable in the weak operator topology. If $\mathcal{X}$ is finite dimensional, all of the above-mentioned topologies coincide on $\mathcal{L}(\mathcal{X})$.*

It is important to know what form the continuous linear functionals on $\mathcal{L}(\mathcal{H})$ in these various topologies take (cf. [**18**, p. 37] for more detail).

PROPOSITION 1.7. *Suppose $\mathcal{M}$ is a linear manifold in $\mathcal{L}(\mathcal{H})$. Then the WOT-closure and SOT-closure of $\mathcal{M}$ coincide. The WOT-continuous linear functionals on $\mathcal{M}$ coincide with the SOT-continuous linear functionals on $\mathcal{M}$, and these are exactly the linear functionals of the form*

$$f(T) = \sum_{i=1}^{n} (Tx_i, y_i) = \operatorname{tr}\left(T\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\right), \qquad T \in \mathcal{M},$$

where $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ are any equipotent finite sequences from $\mathcal{H}$. The weak-continuous linear functionals on $\mathcal{M}$ are exactly those of the form*

$$g(T) = \sum_{i=1}^{\infty} (Tx_i, y_i) = \operatorname{tr}\left(T\left(\sum_{i=1}^{\infty} x_i \otimes y_i\right)\right), \qquad T \in \mathcal{M},$$

where $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ are any square-summable sequences from $\mathcal{H}$. Moreover, the sequences $\{x_i\}$ and $\{y_i\}$ may be chosen to satisfy $\|g\| = \Sigma\|x_i\|^2 = \Sigma\|y_i\|^2$.

We come now to the subject of these lectures.

DEFINITION 1.8. With $\mathcal{X}$ as always, a *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{X})$ that contains $1_{\mathcal{X}}$ and is closed in the weak* topology on $\mathcal{L}(\mathcal{X})$.

EXAMPLES 1.9. The following are dual algebras:

(a) $\mathcal{L}(\mathcal{X})$, where $1 \leqslant \dim \mathcal{X} \leqslant \aleph_0$;

(b) any finite-dimensional subalgebra of $\mathcal{L}(\mathcal{X})$ that contains $1_{\mathcal{X}}$, for example, the algebra of scalar operators;

(c) any von Neumann algebra on $\mathcal{H}$ which contains $1_{\mathcal{H}}$;

(d) the *commutant* $\mathcal{S}'$ of any subset $\mathcal{S}$ of $\mathcal{L}(\mathcal{X})$;

(e) the smallest weak*-closed subalgebra $\mathscr{A}_T$ of $\mathscr{L}(\mathscr{X})$ that contains a given operator T and $1_{\mathscr{X}}$. The dual algebra $\mathscr{A}_T$ is called the *dual algebra generated by* T;

(f) the algebra of all analytic Toeplitz operators relative to some fixed orthonormal basis for $\mathscr{H}$. This is a particular case of (e).

EXAMPLE 1.10. There exists a commutative dual algebra that is not closed in the WOT. Indeed, let K be an operator in $\mathscr{C}_1(\mathscr{H})$ that is not of finite rank, and let $\mathscr{M}$ denote the kernel of the weak*-continuous linear functional induced on $\mathscr{L}(\mathscr{H})$ by K, i.e.,

$$\mathscr{M} = \{T \in \mathscr{L}(\mathscr{H}) : \operatorname{tr}(KT) = 0\}.$$

Note that $\mathscr{M}$ is weak*-closed but not WOT-closed. (For, if $\mathscr{M}$ were WOT-closed, then K would induce a WOT-continuous functional on $\mathscr{M}$, and thus by Proposition 1.7, K would necessarily be of finite rank.) Now consider the algebra $\mathscr{A} \subset \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ defined by

$$\mathscr{A} = \left\{ \begin{pmatrix} \lambda & T \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbf{C},\ T \in \mathscr{M} \right\}.$$

It is easy to see that $\mathscr{A}$ is weak*-closed but not WOT-closed.

PROBLEM 1.11. Does there exist a singly generated dual algebra $\mathscr{A}_T$ that is not WOT-closed? Westwood [33a] has shown that there exists a (WOT-closed) singly generated dual algebra on which the weak* topology and WOT do not coincide.

DEFINITION 1.12. If $\mathscr{S}$ is any subset of $\mathscr{L}(\mathscr{X})$, then ${}^{\perp}\mathscr{S} = \{K \in \mathscr{C}_1 : \langle K, S\rangle = 0,\ S \in \mathscr{S}\}$, is the *preannihilator* of $\mathscr{S}$, which is a (weakly closed) subspace of $\mathscr{C}_1$. (Here we mention for the first time the *weak topology* on $\mathscr{C}_1$ that accrues to it as the predual of $\mathscr{L}(\mathscr{X})$. We also note that in these lectures the word "subspace" always means "norm-closed linear manifold".)

DEFINITION 1.13. If $\mathscr{S}$ is a subset of $\mathscr{L}(\mathscr{X})$, then a subspace $\mathscr{M}$ of $\mathscr{X}$ is a nontrivial invariant subspace (n.i.s.) for $\mathscr{S}$ if $(0) \neq \mathscr{M} \neq \mathscr{X}$ and $S\mathscr{M} \subset \mathscr{M}$ for every $S \in \mathscr{S}$ and a *nontrivial hyperinvariant subspace* (n.h.s.) for $\mathscr{S}$ if it is a n.i.s. for the dual algebra $\mathscr{S}'$. The lattice of all invariant subspaces for $\mathscr{S}$ will be denoted by $\operatorname{Lat}(\mathscr{S})$ and the lattice of all hyperinvariant subspaces of $\mathscr{S}$ by $\operatorname{Hlat}(\mathscr{S})$.

The following elementary proposition shows the importance of preannihilators and rank-one operators.

PROPOSITION 1.14. *A dual algebra $\mathscr{A} \subset \mathscr{L}(\mathscr{X})$ has a n.i.s. if and only if ${}^{\perp}\mathscr{A}$ contains some rank-one operator $x \otimes y$, and a n.h.s. if and only if ${}^{\perp}(\mathscr{A}')$ contains a rank-one operator.*

DEFINITION 1.15. If $\mathscr{S}$ is a subset of $\mathscr{L}(\mathscr{X})$ we write $\operatorname{AlgLat}(\mathscr{S})$ for the dual algebra (which is WOT-closed) consisting of all $T \in \mathscr{L}(\mathscr{X})$ with $\operatorname{Lat}(T) \supset \operatorname{Lat}(\mathscr{S})$. A (necessarily WOT-closed) subalgebra $\mathscr{B}$ of $\mathscr{L}(\mathscr{X})$ is *reflexive* if

$\mathscr{B} = \operatorname{Alg}\operatorname{Lat}(\mathscr{B})$. An operator T in $\mathscr{L}(\mathscr{X})$ is *reflexive* if $\mathscr{W}_T$, the smallest WOT-closed algebra containing T and $1_{\mathscr{X}}$, is reflexive.

Whether an algebra $\mathscr{B}$ or an operator T is reflexive is of considerable interest, because reflexive algebras and operators, as one can easily see from the definitions, have a good supply of nontrivial invariant subspaces. This topic is studied in Chapter IX.

The following elementary facts about preannihilators are extracted from [26a].

PROPOSITION 1.16. *If $\mathscr{M}$ is a weak*-closed subspace of $\mathscr{L}(\mathscr{H})$, then $\mathscr{M}$ is WOT-closed if and only if ${}^{\perp}\mathscr{M}$ is the $\|\ \|_1$-closure of the finite-rank operators in ${}^{\perp}\mathscr{M}$.*

PROPOSITION 1.17. *If $\mathscr{A} \subset \mathscr{L}(\mathscr{X})$ is a dual algebra, then $\mathscr{A}$ is reflexive if and only if ${}^{\perp}\mathscr{A}$ is the smallest subspace of $\mathscr{C}_1$ containing all the rank-one operators in ${}^{\perp}\mathscr{A}$.*

PROPOSITION 1.18. *If $\mathscr{S}$ is a subset of $\mathscr{L}(\mathscr{H})$ then $\mathscr{S}'$ is a dual algebra whose preannihilator ${}^{\perp}(\mathscr{S}')$ is the subspace of $\mathscr{C}_1$ generated by the set $\{KS - SK: K \in \mathscr{C}_1,\ S \in \mathscr{S}\}$. In particular, if $T \in \mathscr{L}(\mathscr{H})$, then ${}^{\perp}(\mathscr{A}'_T)$ is the $\|\ \|_1$-closure of the linear manifold $\{TK - KT: K \in \mathscr{C}_1\}$, and hence T has a n.h.s. if and only if there exists a rank-one operator $x \otimes y$ and a sequence $\{K_n\}_{n=1}^{\infty}$ of operators in $\mathscr{C}_1$ such that $\|(x \otimes y) - (TK_n - K_nT)\|_1 \to 0$.*

We turn now to the proposition which shows why dual algebras are named as they are. For a proof, see [16, Proposition 2.1 and Corollary 2.2].

PROPOSITION 1.19. *Let X be a complex Banach space and let $\mathscr{M}$ be a weak* closed subspace of X^* with preannihilator ${}^{\perp}\mathscr{M}$. Then $X/{}^{\perp}\mathscr{M}$ is a Banach space whose dual $(X/{}^{\perp}\mathscr{M})^*$ can be identified with $\mathscr{M}$. In particular, with $\mathscr{X}$ as always, if $\mathscr{M}$ is a weak* closed linear manifold in $\mathscr{L}(\mathscr{X})$, then $\mathscr{C}_1(\mathscr{X})/{}^{\perp}\mathscr{M} = Q_{\mathscr{M}}$ is a Banach space whose dual space can be identified with $\mathscr{M}$. Under this identification the pairing between $\mathscr{M}$ and $Q_{\mathscr{M}}$ is given by the bilinear functional $\langle T, [L]\rangle =$ $\operatorname{tr}(TL)$, $T \in \mathscr{M}$, $[L] \in Q_{\mathscr{M}}$, where, as usual, we write $[L]$ for the coset in $Q_{\mathscr{M}}$ of an element $L \in \mathscr{C}_1(\mathscr{X})$.*

Henceforth in these lectures we will routinely make the identification of $Q^*_{\mathscr{M}}$ and $\mathscr{M}$ without further discussion.

The following theorem is frequently used in what follows. For a proof, see [16, §2].

THEOREM 1.20. *If $\mathscr{M}$ is a linear manifold in $\mathscr{L}(\mathscr{H})$, then $\mathscr{M}$ is weak*-closed if and only if $\mathscr{M}$ intersects the closed unit ball in $\mathscr{L}(\mathscr{H})$ in a weak*-closed set. If $\mathscr{M}$ is weak*-closed, then the weak* topology that accrues to $\mathscr{M}$ as the dual space of $Q_{\mathscr{M}}$ coincides with the relative weak* topology that accrues to $\mathscr{M}$ as a subspace of $\mathscr{L}(\mathscr{H})$. Furthermore, if $\mathscr{M}$ is weak*-closed, X is a separable complex Banach space, and $\Phi: X^* \to \mathscr{M}$ is a linear mapping, then Φ is continuous when X^* and $\mathscr{M}$ are given their weak* topologies if and only if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X^**

that is weak-convergent to 0, then* $\{\Phi(x_n)\}$ *is weak* convergent to 0 in* $\mathcal{M}$. *Finally, if* Φ *is weak* continuous and has range weak* dense in* $\mathcal{M}$, *then there exists a bounded, one-to-one, linear map* $\phi: Q_{\mathcal{M}} \to X$ *such that* $\Phi = \phi^*$, *and if* Φ *has trivial kernel and norm-closed range, then* $\Phi(X^*) = \mathcal{M}$, Φ *is a weak* homeomorphism of* X^* *onto* $\mathcal{M}$, *and* $\phi: Q_{\mathcal{M}} \to X$ *is invertible.*

We shall also need frequently in what follows the concept of the absolutely convex hull of a set E in a complex vector space X. The set E is said to be *balanced* if $\lambda E \subset E$ for all complex numbers λ satisfying $|\lambda| \leqslant 1$. The *absolutely convex hull* of E, denoted by $\operatorname{aco}(E)$, is the smallest convex and balanced set containing E. Alternatively it is the collection of all linear combinations $\alpha_1 x_1 + \cdots + \alpha_n x_n$ of vectors $x_1, \cdots, x_n$ in E such that $|\alpha_1| + \cdots + |\alpha_n| \leqslant 1$. If X is a Banach space, then the *closed absolutely convex hull* of E, denoted by $\overline{\operatorname{aco}}(E)$, is the norm-closure of $\operatorname{aco}(E)$. The following proposition, whose proof may be found in [**17**, Proposition 2.2], will be quite useful later.

PROPOSITION 1.21. *Let X be a complex Banach space, let M be a positive number, and let E be a subset of X. Then*

$$\|\phi\| \leqslant M \sup_{x \in E} |\phi(x)|, \qquad \phi \in X^*,$$

if and only if $\overline{\operatorname{aco}}(E)$ contains the closed ball of radius $1/M$ about the origin in X. In particular, if E is a subset of the closed unit ball in X and

$$\|\phi\| = \sup_{x \in E} |\phi(x)|, \qquad \phi \in X^*,$$

then $\overline{\operatorname{aco}}(E)$ is the closed unit ball in X.

II. Simultaneous Systems of Equations in the Predual of a Dual Algebra

Let $\mathcal{H}$ be any separable, complex Hilbert space, and let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra with predual $Q_{\mathcal{A}}$. If x and y are vectors in $\mathcal{H}$, then the associated rank-one operator $x \otimes y$ belongs to $\mathcal{C}_1(\mathcal{H})$, and, as noted above, we denote by $[x \otimes y]_{Q_{\mathcal{A}}}$, or $[x \otimes y]$ when there is no possibility of confusion, the image of $x \otimes y$ in $Q_{\mathcal{A}}$. Since every operator L in $\mathcal{C}_1(\mathcal{H})$ can be written as $L = \sum_{i=1}^{n} x_i \otimes y_i$ for some square summable sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ (with convergence in the norm $\| \ \|_1$), it follows immediately that every element of $Q_{\mathcal{A}}$ has the form $\sum_{i=1}^{\infty}[x_i \otimes y_i]$. The first new idea in the sequence of developments under consideration is due to Scott Brown [15], who showed that for certain subnormal operators T, the dual algebra $\mathcal{A}_T$ generated by T has the property that its predual $Q_T = Q_{\mathcal{A}_T}$ consists entirely of elements of the form $[L] = [x \otimes y]$ with x and y nonzero. Brown's remarkable new idea led to proofs, over the past five years, that other preduals Q_T corresponding to T in various classes of operators have this same "rank-one" structure, and a number of new invariant-subspace theorems have resulted. The following definition will play a central role throughout the book. Although the concepts introduced will mostly be studied in the context of dual algebras, in several places the more general setting of weak*-closed subspaces is appropriate.

DEFINITION 2.01. Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a weak*-closed subspace, and let n be any cardinal number such that $1 \leqslant n \leqslant \aleph_0$. Then $\mathcal{M}$ will be said to have property $(\mathbf{A}_n)$ provided every $n \times n$ system of simultaneous equations of the form

$$(2\alpha) \qquad [x_i \otimes y_j] = [L_{ij}], \qquad 0 \leqslant i, j < n,$$

(where the $[L_{ij}]$ are arbitrary but fixed elements from $Q_{\mathcal{M}}$) has a solution $\{x_i\}_{0 \leqslant i < n}$, $\{y_i\}_{0 \leqslant i < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. Furthermore if $n \in \mathbb{N}$ and r is a fixed real number satisfying $r \geqslant 1$, then a weak*-closed subspace $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ (with property $\mathbf{A}_n$) is said to have property $(\mathbf{A}_n(r))$ if for every $s > r$ and every $n \times n$ array $\{[L_{ij}]\}_{0 \leqslant i,j < n}$ from $Q_{\mathcal{M}}$, there exist sequences $\{x_i\}_{0\leqslant i<n}$, $\{y_j\}_{0\leqslant j<n}$ that satisfy (2α) and also satisfy the following conditions:

$$(2\beta)\qquad \begin{aligned} \|x_i\| &\leqslant \Big(s\sum_{0\leqslant j<n}\|[L_{i,j}]\|\Big)^{1/2}, \qquad 0\leqslant i<n,\\ \|y_j\| &\leqslant \Big(s\sum_{0\leqslant i<n}\|[L_{i,j}]\|\Big)^{1/2}, \qquad 0\leqslant j<n. \end{aligned}$$

Finally, a weak*-closed subspace $\mathscr{M}\subset\mathscr{L}(\mathscr{X})$ has property $(\mathbf{A}_{\aleph_0}(r))$ (for some real $r\geqslant 1$) if for every $s>r$ and every array $\{[L_{ij}]\}_{i,j=0}^{\infty}$ from $Q_{\mathscr{M}}$ with summable rows and columns, there exist sequences $\{x_i\}_{i=0}^{\infty}$ and $\{y_j\}_{j=0}^{\infty}$ from $\mathscr{X}$ that satisfy (2α) and (2β) (with $n=\aleph_0$). (Note that it is not immediate that a dual algebra with property $(\mathbf{A}_{\aleph_0}(r))$ has property $(\mathbf{A}_{\aleph_0})$, but this is proved in Theorem 3.7.)

DEFINITION 2.02. If $\mathscr{M}\subset\mathscr{L}(\mathscr{X})$ is a weak*-closed subspace and n is a positive integer, then $\mathscr{M}$ will be said to have property $(\mathbf{A}_{1/n})$ if every $[L]$ in $Q_{\mathscr{M}}$ can be written in the form

$$(2\gamma)\qquad [L]=\sum_{i=1}^{n}[x_i\otimes y_i]$$

for certain sequences of vectors $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_n\}$ (depending on $[L]$). If $\mathscr{M}$ has property $(\mathbf{A}_{1/n})$, $r\geqslant 1$, and for every $s>r$, the sequences $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_n\}$ satisfying (2γ) can also be chosen to satisfy

$$(2\delta)\qquad \sum_{i=1}^{n}\|x_i\|^2\leqslant s\|[L]\|, \qquad \sum_{i=1}^{n}\|y_i\|^2\leqslant s\|[L]\|,$$

then $\mathscr{M}$ will be said to have property $(\mathbf{A}_{1/n}(r))$.

It is obvious that the properties in the two-way infinite sequence

$$\{\dots,(\mathbf{A}_{1/n}),\dots,(\mathbf{A}_{1/2}),(\mathbf{A}_1),(\mathbf{A}_2),\dots,(\mathbf{A}_n),\dots,(\mathbf{A}_{\aleph_0})\}$$

become successively stronger as one moves to the right in the sequence, and it is also obvious that all of these properties, along with the properties $(\mathbf{A}_n(r))$, are unitary invariants for dual algebras.

EXAMPLE 2.03. It is clear that $\mathscr{L}(\mathscr{H})$ does not have any of the properties $(\mathbf{A}_{1/n})$. Furthermore, the commutative dual algebra $\mathscr{A}\subset\mathscr{L}(\mathscr{H}\oplus\mathscr{H})$ in Example 1.10 does not have any of the properties $(\mathbf{A}_{1/n})$, and thus certainly has none of the properties $(\mathbf{A}_n)$. To see this, fix $n\in\mathbf{N}$, and let F be an operator of rank $n+1$ in $\mathscr{L}(\mathscr{H})$. We define a WOT-continuous linear functional ϕ_F on $\mathscr{A}$ by setting $\phi_F(A)=\operatorname{tr}(FT)$, where

$$A=\begin{pmatrix}\lambda & T\\ 0 & \lambda\end{pmatrix},$$

(and $T \in \mathscr{M}$). Since $\mathscr{M}$ is the kernel of a weak* continuous linear functional on $\mathscr{L}(\mathscr{H})$ that is not WOT-continuous (cf. Example 1.10), $\mathscr{M}$ is WOT-dense in $\mathscr{L}(\mathscr{H})$ ([13, Problem 140]). If one supposes that $\mathscr{A}$ has property $(\mathbf{A}_{1/n})$, then there exist vectors $\tilde{x}_1, \ldots, \tilde{x}_n$ and $\tilde{y}_1, \ldots, \tilde{y}_n$ in $\mathscr{H} \oplus \mathscr{H}$ such that

$$\phi_F(A) = \sum_{i=1}^{n} (A\tilde{x}_i, \tilde{y}_i), \qquad A \in \mathscr{A},$$

and a short calculation leads to the contradiction that F has rank less than or equal to n. Thus the example is established.

Certain elementary consequences result when a dual algebra has one of the properties $(\mathbf{A}_n)$.

PROPOSITION 2.04. *If $\mathscr{M}$ is any weak*-closed subspace with property $(\mathbf{A}_n)$ for some $1 \leqslant n \leqslant \aleph_0$ [resp., property $(\mathbf{A}_{1/n})$ for some $n \in \mathbf{N}$], and $\mathscr{N}$ is any weak*-closed subspace of $\mathscr{M}$, then $\mathscr{N}$ has the same property. Furthermore, if $\mathscr{M}$ has property $(\mathbf{A}_n(r))$ for some $1 \leqslant n \leqslant \aleph_0$ and $r \geqslant 1$ [resp., property $(\mathbf{A}_{1/n}(r))$ for some $n \in \mathbf{N}$ and $r \geqslant 1$], then $\mathscr{N}$ has the same property.*

PROOF. If $[L] \in Q_{\mathscr{N}}$, then $[L]$ induces a weak*-continuous linear functional ϕ on $\mathscr{N}$, and ϕ can be extended to a weak*-continuous linear functional ϕ' on $\mathscr{M}$ (cf. [13, Proposition 14.13]). Thus ϕ' is induced by an element $[L']$ in $Q_{\mathscr{M}}$, and the first two statements of the proposition follow easily. To prove the second two statements, one must know that ϕ' may be chosen to have norm arbitrarily close to that of ϕ, and that this is the case is proved in [23, Lemma 2.4].

As far as we know, the earliest mention in the literature of properties $(\mathbf{A}_2, \ldots, (\mathbf{A}_{\aleph_0})$ is in [9–11], and we know of no references to the properties $(\mathbf{A}_{1/n})$, $n > 1$, although the idea is natural enough. On the other hand, properties $(\mathbf{A}_1)$ and $(\mathbf{A}_1(r))$ have been studied somewhat, perhaps most generally in [23], where they were called properties "D_σ" and "$D_\sigma(r)$", respectively. Surely one of the oldest results in this area is the following, half of which is proved in [26b] and the other half of which follows from the polar decomposition for weak* continuous linear functionals on a von Neumann algebra and [18, p. 233].

PROPOSITION 2.05. *A von Neumann subalgebra of $\mathscr{L}(\mathscr{H})$ has property $(\mathbf{A}_1)$ if and only if it has a separating vector, and has property $(\mathbf{A}_1(1))$ if it has property $(\mathbf{A}_1)$.*

Here are some additional nice results about dual algebras with property $(\mathbf{A}_1)$ from [23].

PROPOSITION 2.055. *A direct sum (or direct integral) of dual algebras, each of which has property $(\mathbf{A}_1(r))$, also has property $(\mathbf{A}_1(r))$. If the direct sum $\mathscr{A} = \sum_{i=1}^{\infty} \oplus \mathscr{A}_i$ of an infinite sequence of dual algebras $\mathscr{A}_i$ has property $(\mathbf{A}_1)$, and each $\mathscr{A}_i$ has some property $(\mathbf{A}_1(r_i))$, then $\mathscr{A}$ has property $(\mathbf{A}_1(r))$ for some fixed $r \geqslant 1$.*

THEOREM 2.06. *If $\mathscr{A} \subset \mathscr{L}(\mathscr{X})$ is a dual algebra and there exists a vector $x \in \mathscr{X}$ which is separating for $\mathscr{A}$ and has the property that $\mathscr{A}x$ is a (closed) subspace of $\mathscr{X}$, then $\mathscr{A}$ has property $(\mathbf{A}_1(r))$ for some $r \geqslant 1$. Consequently if T acts on a finite-dimensional space or is an algebraic operator in $\mathscr{L}(\mathscr{H})$, then $\mathscr{A}_T$ has property $(\mathbf{A}_1(r))$ for some $r \geqslant 1$.*

PROPOSITION 2.065. *If $T \in \mathscr{L}(\mathscr{X})$ and satisfies a polynomial equation of degree two, or is a two-normal operator, then $\mathscr{A}_T$ has property $(\mathbf{A}_1(\sqrt{10}))$. However, there is a 4-normal operator that does not have property $(\mathbf{A}_1)$.*

The last statement of Proposition 2.065 is quite intriguing, and results from the following important example and Proposition 2.055.

EXAMPLE 2.07 [**23**, Theorem 5.9]. Consider the sequence of operators

$$T_n = \begin{pmatrix} 0 & n & 0 & 0 \\ 0 & 0 & n^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad n \in \mathbf{N},$$

acting on $\mathbf{C}^{(4)}$. It follows from Proposition 2.06 that for each $n \in \mathbf{N}$, the dual algebra $\mathscr{A}_{T_n}$ has property $(\mathbf{A}_1(r_n))$ for some $r_n \geqslant 1$. We show, however, that there exists no fixed r such that $\mathscr{A}_{T_n}$ has property $(\mathbf{A}_1(r))$ for each $n \in \mathbf{N}$. Suppose, to the contrary, that such an r exists. For $n \in \mathbf{N}$, consider the linear functional ϕ_n on $\mathscr{A}_{T_n}$ defined by setting $\phi_n((a_{ij})) = a_{23} + a_{14}$ for $(a_{ij}) \in \mathscr{A}_{T_n}$. Obviously, $\|\phi_n\| \leqslant 2$ for all n. There exist vectors f_n, g_n in $\mathbf{C}^{(4)}$ such that $\phi_n(A) = (Af_n, g_n)$ for all A in $\mathscr{A}_{T_n}$. Moreover, by using the proper scale factor, we may suppose that $\|f_n\| = 1$ and $\|g_n\| \leqslant (2+\varepsilon)r \leqslant 3r$. By writing $f_n = (s_n, t_n, u_n, v_n)$ and $g_n = (\bar{w}_n, \bar{x}_n, \bar{y}_n, \bar{z}_n)$, and computing $\phi_n(T_n^k)$ for $k = 1, 2, 3$, we obtain, for $n \in \mathbf{N}$,

$$nt_n w_n + n^2 u_n x_n + v_n y_n = n^2,$$

$$n^3 u_n w_n + n^2 v_n x_n = 0,$$

$$n^3 v_n w_n = n^3,$$

$$|s_n|^2 + |t_n|^2 + |u_n|^2 + |v_n|^2 = 1,$$

$$|w_n|^2 + |x_n|^2 + |y_n|^2 + |z_n|^2 \leqslant 9r^2,$$

and these equations are easily seen not to be compatible for sufficiently large n. Thus we have reached a contradiction.

The reader who has been paying close attention has perceived that we do not know the answer to the following question.

PROBLEM 2.075. If a dual algebra $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ has property $(\mathbf{A}_1)$, then does it necessarily have property $(\mathbf{A}_1(r))$ for some $r \geqslant 1$? (Examples are known of surjective bilinear maps that are not open.)

It is also important to point out that Olin and Thompson, making full use of the pioneering work of S. Brown in [**15**], proved in [**27**] the following.