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Sobolev Spaces of Infinite Order and Differential Equations

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Sobolev Spaces of Infinite Order and Differential Equations

Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R. van Gulik's *The Chinese Maze Murders*.

It isn't that they can't see the solution. It is that they can't see the problem.

G.K. Chesterton. *The Scandal of Father Brown* 'The point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "experimental mathematics", "CFD", "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics. This programme, Mathematics and Its Applications, is devoted to new emerging (sub)disciplines and to such (new) interrelations as exempla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

Because of the wealth of scholarly research being undertaken in the Soviet Union, Eastern Europe, and Japan, it was decided to devote special attention to the work emanating from these particular regions. Thus it was decided to start three regional series under the umbrella of the main MIA programme.

There are good reasons to believe that in the future, looking back, the 20-th century will be regarded as the age of functional analysis or, more generally the period in which various stings were taken out of infinity. Thus we have, by and large, learned now to live with infinite dimensional (function) spaces and operators and functionals on them. There are however other finiteness aspects which should probably be removed both in view of applications and for greater power and elegance of theory. One such is that it is (historically) customary to restrict attention to differential operators of finite order. In this book, differential operators of infinite order are considered and the Sobolev space theory needed to live with them is developed and applied. It is probably unnecessary to argue that differential operators are important and occur naturally in many places. There is an increasing algebraic and formal manipulation aspect to analysis at the moment, and in my personal opinion ∞-order differential operators will among others play an important role in putting formal algebraic arguments with differential operators (and their inverses) on a sound footing.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you know of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange.

Bussum, July 1985

Michiel Hazewinkel

To my father -
a schoolteacher
of mathematics

PREFACE

The present book is devoted to the study of boundary value problems of infinite order and the corresponding functional spaces. Despite the fact that the first results in this direction were obtained 10 years ago (see Bibl.), one can now suggest rather good foundations (of course, in our opinion) for the theory of both topics mentioned above.

Before each Chapter there is an introduction, so we do not describe here the contents in detail, but give only their general character. From this point of view the material of our book may be divided into two main parts:

1) the theory of boundary value problems of infinite order itself (Ch. II, III, VI);

2) the theory of Sobolev spaces of infinite order $W^{\infty}\{a_{\alpha}, p_{\alpha}\}$ which are the "energy" spaces of the corresponding problems (Ch. I, III - V).

Two theories are of primary importance in the development of the questions described in this book: the theory of nonlinear boundary value problems of finite order and the theory of classical Sobolev spaces W_p^m.

As is known, at present these two theories are essentially two sides of one theory of differential equations of finite order in the corresponding functional spaces. This is also true for the differential equations of infinite order. Moreover, in this case the connection between the boundary value problems and their functional spaces is deeper, since the existence (nontriviality) of energy spaces $W^{\infty}\{a_{\alpha}, p_{\alpha}\}$ itself is, essentially, equivalent to the correctness of the corresponding boundary value problems. In this connection we start with the study of the question of nontriviality of the necessary spaces. The settlement of this question of nontriviality allows for the investigation of not only the corresponding boundary value problems, but also a series of related functional problems: imbedding theory of $W^{\infty}\{a_{\alpha}, p\}$, trace theory of functions $u(x) \in W^{\infty}\ a_{\alpha}, p_{\alpha}$, geometrical characteristics of these spaces etc., all of which are of independent interest.

The present book includes, in particular, the systematic consideration of these questions.

Moscow Energy Institute

Moscow 1984

CONTENTS

CHAPTER I

NONTRIVIALITY OF SOBOLEV SPACES OF INFINITE ORDER

Introduction

In this Chapter we introduce some functional spaces $W^{\infty}\{a_\alpha, p_\alpha\}$ the "metric" of which is determined by the series

$$\varrho(u) \equiv \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_{r_\alpha}^{p_\alpha} < \infty ,$$

where $a_\alpha \geq 0$, $p_\alpha \geq 1$, $r_\alpha \geq 1$ are arbitrary sequences of numbers; $\|\cdot\|_r$ is the norm in Lebesgue space L_r.

We call such spaces the Sobolev spaces of infinite order. Infinitely differentiable functions $u(x): G \to \mathbb{C}^1$ $(G \subset \mathbb{R}^n)$ are the elements of these spaces; moreover, the functions $u(x)$ may satisfy some boundary conditions.

In contrast with the finite order Sobolev spaces, the very first question, which arises in the study of the spaces $W^{\infty}\{a_\alpha, p_\alpha\}$, is the question of their nontriviality (or nonemptiness), i. e. the question of the existence of a function $u(x)$ such that $\varrho(u) < \infty$

It turns out that the answer of this question depends not only on the given parameters a_α, p_α of the space $W^{\infty}\{a_\alpha, p_\alpha\}$, but also on the region G (the domain of definition of the functions $u(x) \in W^{\infty}\{a_\alpha, p_\alpha\}$), boundary values of the functions $u(x)$ etc.

In §§ 1 - 3 we give necessary and sufficient conditions for nontriviality of Sobolev spaces of infinite order in three of the most commonly encountered cases in analysis: a bounded region $G \subset \mathbb{R}^n$, full Euclidean space $\mathbb{R}^n$ and the torus $T^n = S^1 \times \ldots \times S^1$, where S^1 is the unit circle. In § 4 the case of the strip is considered.

In the case of a bounded region $G \subset \mathbb{R}^n$ the question of nontriviality of $\mathring{W}^{\infty}\{a_\alpha, p_\alpha\}$ turns out to be closely related to the classical theory of Hadamard's quasianalytic classes $C\{M_N\}$. Namely, the space $\mathring{W}^{\infty}\{a_\alpha, p_\alpha\}$ is nontrivial precisely when a certain sequence $M_N > 0$, which is uniquely defined by the numbers a_α and p_α for $|\alpha| = N$, generates a one-dimensional non-quasianalytic class $C\{M_N\}$.

In the cases of the torus and full Euclidean space $\mathbb{R}^n$ the nontriviality of the space $W^{\infty}\{a_\alpha, p_\alpha\}$ is connected with the characteristic function of this space

$$\varphi(\xi) = \sum_{|\alpha|=0}^{\infty} a_\alpha \xi^{\alpha p_\alpha}, \quad \xi = (\xi_1, \dots, \xi_n).$$

It is very simple to formulate the criterion of nontriviality of $W^\infty\{a_\alpha, p_\alpha\}$ in the case of a bounded sequence p_α and, in particular, in the Banach case $p_\alpha \equiv p$. Namely, if $p_\alpha \leq$ const, then $W^\infty\{a_\alpha, p_\alpha\}(T^n)$ is nontrivial if and only if the function

$$a(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha, \quad z \in \mathbb{C}^n,$$

is an entire function.

Under the same conditions the nontriviality of the space $W^\infty\{a_\alpha, p_\alpha\}(\mathbb{R}^n)$ of functions $u(x)\colon \mathbb{R}^n \to \mathbb{C}^1$ is equivalent to the analyticity of the function $a(z)$ in a neighbourhood of zero.

In the case of the strip $G = [0,a] \times \mathbb{R}^n$ the criterion for nontriviality of the space $\mathring{W}^\infty\{a_{n\alpha}, p_\alpha\}$ is, roughly speaking, a combination of the criterion for nontriviality of the space $\mathring{W}^\infty$ on the interval $(0,a)$ and the criterion for nontriviality of the space W^∞ in the full Euclidean space.

§ 1. Criterion for nontriviality of the spaces $\mathring{W}^\infty\{a_\alpha, p_\alpha\}$ in the case of a bounded domain

Let $G \subset \mathbb{R}^n$ be a bounded domain and let Γ be the boundary of this domain. Let us denote the space of infinitely differentiable functions $u(x)\colon G \to \mathbb{C}^1$ such that $D^\omega u|_\Gamma = 0$, $|\omega| = 0, 1, \dots$, by $C_0^\infty(G)$. (Here $\omega = (\omega_1, \dots, \omega_n)$, $\omega_j \in \mathbb{N}$, $1 \leq j \leq n$; D^ω is the standard notation

$$D^\omega = \frac{\partial^{|\omega|}}{\partial x_1^{\omega_1} \dots \partial x_n^{\omega_n}}, \quad |\omega| = \omega_1 + \dots + \omega_n;$$

in the same way D^α, D^γ etc.). In other words

$$C_0^\infty(G) = \left\{ u(x) \in C^\infty(G)\colon D^\omega u|_\Gamma = 0, \ |\omega| = 0, 1, \dots \right\}.$$

One can suppose that $u(x) \equiv 0$ for $x \bar{\in} G$. We shall refer to such a function as a finite function.

Let us consider the following functional space

$$\mathring{W}^{\infty}\{a_\alpha,p_\alpha\} = \left\{u(x)\in C_0^{\infty}(G):\ \varrho(u) \equiv \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u\|_{r_\alpha}^{p_\alpha} < \infty\right\},$$

where $a_\alpha \geq 0$, $p_\alpha \geq 1$, $r_\alpha \geq 1$ are arbitrary sequences of numbers; $\|\cdot\|_r$ is the norm in $L_r(G)$.[1)]

It is clear that the question of nontriviality of $\mathring{W}^{\infty}\{a_\alpha,p_\alpha\}$ arises if among the numbers $a_\alpha \geq 0$ there are infinitely many greater than zero.

Definition 1.1. The space $\mathring{W}^{\infty}\{a_\alpha,p_\alpha\}$ is called nontrivial if it contains at least one function which not identically equal to zero, i. e. there is a function $u(x)\in C_0^{\infty}(G)$ such that $\varrho(u)<\infty$.

Before formulating a nontriviality criterion let us introduce the following number sequence. Namely, let M_N, $N = 0,1,\ldots$, be the solution of the equation

$$\sum_{|\alpha|=N} a_\alpha M_N^{p_\alpha} = 1 \tag{1.1}$$

with $M_N = +\infty$ if $a_\alpha \equiv 0$ for all $|\alpha| = N$. (Obviously, the relations (1.1) define the numbers M_N uniquely.)

Theorem 1.1. The space $\mathring{W}^{\infty}\{a_\alpha,p_\alpha\}$ is nontrivial if and only if the sequence M_N, $N = 0,1,\ldots$, defines a non-quasianalytic Hadamard's class of one real variable.

Remark. Let us recall the definition of Hadamard's class $C\{M_N\}$, where M_N, $N = 0,1,\ldots$, is a number sequence. Namely,

$$C\{M_N\} = \left\{u(x)\in C^{\infty}(a,b):\ |D^N u(x)| \leq KM_N,\quad x\in(a,b),\quad N = 0,1,\ldots\right\},$$

where $K > 0$ is a constant, depending (in general) on the function $u(x)$. The class $C\{M_N\}$ is called quasianalytic when the following condition is valid:

if $u(x)\in C\{M_N\}$, $v(x)\in C\{M_N\}$ and $D^N u(x_0) = D^N v(x_0)$, $N = 0,1,\ldots,$

($x_0 \in (a,b)$ is a fixed point), then $u(x) \equiv v(x)$ for all $x\in(a,b)$. Otherwise the class $C\{M_N\}$ is called non-quasianalytic.

[1)] As one can see below, the values r_α are immaterial in the question of nontriviality. In view of this fact we don't use r_α in the notation.

The various algebraic conditions (criteria) of non-quasianalyticity are well known (see, for example, S. MANDELBROJT [1]). One of them - the Mandelbrojt-Bang criterion - will be used for the proof of our theorem.

Proof of the Theorem 1.1. To prove the necessity of our conditions it is obviously enough to prove that if the sequence M_N defines a quasianalytic Hadamard's class $C\{M_N\}$ and $u(x) \in \mathring{W}^{\infty}\{a_\alpha, p_\alpha\}$, then $u(x) \equiv 0$. Indeed, $u(x) \in \mathring{W}^{\infty}\{a_\alpha, p_\alpha\}$ implies $u(x) \in C_0^{\infty}(G)$ and, consequently, for any α and $\xi = (\xi_1, \dots, \xi_n)$ one has the inequality

$$|\xi^\alpha| \cdot |\tilde{u}(\xi)| \leq \|D^\alpha u\|_1 \leq K \|D^\alpha u\|_{r_\alpha}, \qquad (1.2)$$

where $\tilde{u}(\xi)$ is the Fourier transform of $u(x)$ and $K > 0$ is a constant, depending only on the measure of G.

Let $\xi = \eta\theta$, where the vector $\theta = (\theta_1, \dots, \theta_n)$, $|\theta_j| \geq 1$, $1 \leq j \leq n$, is fixed and $\eta \in \mathbf{R}^1$ is arbitrary. Then, using (1.2), we obtain the inequality

$$|\eta|^N |\tilde{u}(\eta\theta)| \leq K \|D^\alpha u\|_{r_\alpha}, \quad |\alpha| = N.$$

Since the natural number N is arbitrary and the function u(x) is finite, we obtain that for any $N \geq 2$

$$(1 + \eta^2)|\eta|^{N-2}|\tilde{u}(\eta\theta)| \leq K\|D^\alpha u\|_{r_\alpha}, \quad |\alpha| = N,$$

where $K > 0$ is a constant.[1)]

Thus,

$$\sum_{|\alpha|=N} a_\alpha \left[(1 + \eta^2)|\eta|^{N-2}|\tilde{u}(\eta\theta)| K^{-1}\right] \leq \sum_{|\alpha|=N} a_\alpha \|D^\alpha u\|_{r_\alpha}^{p_\alpha}.$$

Since $u(x) \in \mathring{W}^{\infty}\{a_\alpha, p_\alpha\}$, one can suppose (without less of generality) that

$$\sum_{|\alpha|=N} a_\alpha \|D^\alpha u\|_{r_\alpha}^{p_\alpha} \leq 1$$

and, consequently, taking into account the definition of sequence M_N, $N = 0, 1, \dots$, we shall have the inequalities

$$(1 + \eta^2)|\eta|^{N-2}|\tilde{u}(\eta\theta)| \leq KM_N, \quad N = 2, 3, \dots$$

---

1) Here and below all constants the values of which are non-principal will be denoted by one letter K.

The latter inequalities mean that

$$|D^{N-2}v(y)| \leq KM_N, \quad N = 2,3,\ldots,$$

where the function $v(y)$, $y \in \mathbb{R}^1$, is the inverse Fourier transform of $\tilde{u}(\eta\theta)$ with respect to variable $\eta \in \mathbb{R}^1$.

These inequalities imply that the function $v(y)$ belongs to the Hadamard class $C\{M_{N+2}\}$ which by assumption (together with $C\{M_N\}$) is quasianalytic. On the other hand, from the Paley-Wiener theorem the function $v(y) \in C_0^\infty(\mathbb{R}^1)$; consequently, $v(y) \equiv 0$. Thus for any line $\xi = \eta\theta$, where $\theta = (\theta_1,\ldots,\theta_n)$, $\theta_j \geq 1$, we obtain that $\tilde{u}(\xi) = 0$, i. e. $\tilde{u}(\xi) \equiv 0$ for all $\xi \in \mathbb{R}^n$. It follows that $u(x) \equiv 0$ in G. The necessity is proved.

<u>Sufficiency</u>. The known lemma on functions with compact support of one real variable plays a fundamental part in this proof (cf. S. MANDELBROJT [1], Ch. IV, Theorem 4.1.IV).

<u>Lemma 1.1</u>. Let $\mu_0 = 1$, $\mu_N > 0$ $(N = 1,2,\ldots)$ be a number sequence satisfying the condition

$$\mu_1 + \mu_2 + \ldots < \frac{a}{3}, \quad a > 0. \tag{1.3}$$

Then there exists a function $v(t) \in C_0^\infty(-a,a)$, $t \in \mathbb{R}^1$, such that:

1. $v(0) = 1$, $D^N v(-a) = D^N v(a) = 0$, $N = 0,1,\ldots;$
2. for any $t \in (-a,a)$

$$|D^N v(t)| \leq (\mu_0\mu_1\ldots\mu_N)^{-1}, \quad N = 0,1,\ldots \tag{1.4}$$

<u>Proof of lemma</u>. We choose the continuous function $v_0(t)$ satisfying the following conditions:

1. $0 \leq v_0(t) \leq 1$, $t \in (-a,a)$;
2. $v_0(t) \equiv 1$, if $t \in (-a/3,a/3)$;
3. $v_0(t) \equiv 0$, if $t \in (-a,-2a/3)$ and $t \in (2a/3,a)$.

Besides let $v_0(t)$ be an even function.

Further we define the sequence of functions $v_m(t)$ by the following recursion formula

$$v_m(t) = \frac{1}{2\mu_m} \int_{t-\mu_m}^{t+\mu_m} v_{m-1}(\tau)d\tau, \quad m = 1,2,\ldots$$

In view of condition (1.3) these functions are finite on the interval $(-a,a)$. It is also clear that the functions $v_m(t)$, $m = 1,2,\ldots$, are even and differentiable at least up to order m.

Let us prove that the sequence $v_m(t)$ (strictly speaking a subsequence) converges to a function $v(t) \in C_0^\infty(-a,a)$ as $m \to \infty$. To establish this fact we shall prove first the following inequality

$$\max_{t\in(-a,a)} |D^m v_m(t)| \leq (\mu_0\mu_1\ldots\mu_m)^{-1}, \quad m = 0,1,\ldots \tag{1.5}$$

In fact, for $m = 0$ the inequality (1.5) is evident. Further, for $n \leq m$ we have

$$D^n v_m(t) = \frac{1}{2\mu_m}\left[D^{n-1} v_{m-1}(t+\mu_m) - D^{n-1} v_{m-1}(t-\mu_m)\right]. \tag{1.6}$$

In particular $(n = m)$, we obtain that

$$\max_{t\in(-a,a)} |D^m v_m(t)| \leq \frac{1}{\mu_m} \max_{t\in(-a,a)} |D^{m-1} v_{m-1}(t)|.$$

Thus, the inequality (1.5) is valid if this inequality is valid for $m-1$ too. So the inequality (1.5) is valid for all m.

Let $n \leq m-1$ be arbitrary now. In this case, using the inequality (1.6), we obtain that for any point $t\in(-a,a)$ there exist points $t_1,\ldots,t_{m-n}$ so that

$$D^n v_m(t) = D^n v_{m-1}(t_1) = \ldots = D^n v_n(t_{m-n}).$$

Consequently, from this and inequality (1.5) we have

$$\max_{t\in(-a,a)} |D^n v_m(t)| \leq \max_{t\in(-a,a)} |D^n v_n(t)| \leq (\mu_0\mu_1\ldots\mu_n)^{-1}. \tag{1.7}$$

From this, using Arzelà'stheorem and a diagonal process, we obtain that there exists a subsequence of sequence $v_m(t)$ (we shall denote it as $v_m(t)$ too) and a function $v(t) \in C_0^\infty(-a,a)$ so that

$$v_m(t) \longrightarrow v(t), \ldots, D^n v_m(t) \longrightarrow D^n v(t), \ldots$$

uniformly for $t\in(-a,a)$.