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AUTOMATIC FEEDBACK CONTROL SYSTEM SYNTHESIS

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PREFACE

During the years since the end of World War II there have been a number of books published on the subject of feedback theory, with particular emphasis on the analysis and design of feedback control systems. The principal concern of all these books has been with the synthesis of linear feedback control systems, with emphasis on the techniques for system design through inspection of the behavior of the open-loop transfer function at real frequencies. Although almost all the books have employed the Laplace transform liberally, the critical reader continually must question the motive for writing the transfer functions in terms of the complex frequency s. In almost every case, the authors go to considerable length to explain the Laplace transform, go to great pains to use the complex frequency in the process of setting up the equations describing the behavior of the various components of the system, and then blithely replace s by $j\omega$ before launching the design of the compensation networks.

It is true that certain writers emphasize the importance of the dynamic performance of the system. Generally, however, the reader is left with the unsavory choice of determining the transient performance by the depressingly tedious process of taking the exact inverse Laplace transform or of attempting to evaluate an approximate inverse transform by one of the many systematized procedures. If dynamic performance is really of importance in the design of servo systems, neither alternative is appealing. In neither case is any logical method presented for the improvement of the transient response. If the transient performance is unsatisfactory, about the best that the designer can do is to start the entire design over again, proceeding in a slightly different manner toward a final system which meets frequency-domain specifications, and hope that the transient response will be improved.

In view of this widespread emphasis on design in the domain of real frequencies, what justification exists for the use of the Laplace transform? Unfortunately, any exploitation of the advantages of the Laplace transform demands familiarity with aspects of electrical engineering which are not second nature to the majority of engineers. In particular, maximum utilization of the Laplace transform as a design tool depends, on the part of the designer, upon a firm knowledge of the relationships between the Laplace transforms and the transient and frequency responses, the fundamental concepts of feedback theory, the techniques of network synthesis, the elements of statistical methods of design, and the general concepts used in the analysis of nonlinear systems.

First, the designer must have a working familiarity with the Laplace transform: indeed, a familiarity which goes beyond the simple formal relationships between the time and frequency functions. It is not sufficient to know the mechanical techniques for evaluating direct and inverse transforms. Rather the designer must be conscious of the relationships between time-domain and frequency-domain characteristics, relationships which depend upon the significance of the positions of the poles and zeros of the transfer functions. He must be able to determine the effects, in both time and frequency domains, of varying these pole and zero positions.

Second, the designer must be familiar with the fundamental concepts of feedback theory. Motivation for the use of closed-loop rather than open-loop systems arises from the possibility, with feedback, of controlling sensitivity, output impedance, and subsidiary transmissions while, at the same time, realizing a specified over-all system function. Feedback theory involves a development of the definitions and significances of these controllable quantities and demonstrates system design methods which permit the required control.

Third, if the servo engineer is to appreciate fully the advantages inherent in the use of the Laplace transform, he must be able to synthesize appropriate compensation networks. The familiar lead, lag, and twin-T networks which have so long served as the building blocks of system design fail to support the structure of design when the specifications become stringent, use of isolation amplifiers is denied, or unusual systems are required (such as mechanical or hydraulic networks). It becomes necessary to be able to synthesize networks with prescribed transfer characteristics.

Fourth, the designer must be familiar with the statistical methods of design if he is to cope with the problems associated with realistic signal inputs. A tremendous amount of research effort is being expended in the direction of improving feedback control systems, particularly those systems in which random noise is a primary deterrent to high system performance. Problems associated with the control of corrupting signals which are essentially random in nature are readily handled only with statistical methods.

Fifth, the designer must be acquainted with the basic techniques available for considering nonlinear systems. He must be able to analyze the effects of unwanted nonlinearities present in the system and to synthesize nonlinearities into the system to improve dynamic performance.

This book represents an attempt to organize and unify the background material in these five fields. Emphasis is on the development of the basic theory, although an attempt is made to illustrate the discussion with examples of practical significance. Unfortunately, however, in a book of reasonable length, it is necessary to place most of the emphasis on the theory and allow the reader to use examples from his own experience as illustrations.

The bulk of the material in the book has been taught in a two-semester graduate servomechanisms course in the Electrical Engineering School at Purdue University. This course has followed the introductory course, which has used as texts sections of Brown and Campbell, "Principles of Servomechanisms," Chestnut and Mayer, "Servomechanisms and Regulating System Design," and Ahrendt and Taplin, "Automatic Feedback Control." The author feels that this book, as a text, logically follows any of the three mentioned above. In addition, a serious effort has been made to present the material in a way which will be of value, not only as a textbook, but also as a reference book for industrial engineers. In the contacts which the author has had with industrial problems, he has been impressed by the growing realization on the part of industrial research groups of the necessity for the maximum utilization of available design theory.

In the preparation of the book, an attempt has been made to make as many of the chapters as possible self-contained. Chapter 1 serves as an introduction and a review of the mathematical background. In teaching, the author has usually omitted parts of this chapter until the need for the material arose later in the course. Chapters 2 and 3 present basic theory; in most universities, the material of Chap. 3 is adequately covered in a course on network synthesis. Chapters 4, 5, and 6 describe the important aspects of design in terms of the Laplace transform. Statistical design theory is described in Chaps. 7 and 8, with the former chapter emphasizing the fundamental concepts and the latter chapter describing certain applications of particular interest to the control-systems engineer. Chapter 9 presents the basic characteristics of sampled-data feedback systems, and the book concludes with the two chapters on the analysis of nonlinear feedback systems.

The author is deeply indebted to a number of individuals who assisted in the preparation of the manuscript. In every instance, the author received encouragement and complete cooperation from the electrical engineering staff at Purdue University. Although it is difficult to name all individuals, discussions with Drs. J. R. Burnett and G. R. Cooper were particularly helpful. The proofreading was completely the work of Mr. T. A. Savo, who also contributed encouragement at every stage of the writing and innumerable constructive criticisms for improvement of the presentation. The author's wife typed the complete manuscript and assisted extensively with the figures, etc. Indeed, the book is in every respect the result of the mutual efforts of the author and his wife.

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CHAPTER 1

INTRODUCTION

During the last decade, feedback systems and active networks have become increasingly important in a number of branches of engineering. Simultaneously, there has been increased interest in the dynamic characteristics of systems, in contrast to the earlier interest in static characteristics. Furthermore, emphasis has been turned toward the performance of systems excited by aperiodic (or transient) signals and signals which can only be described statistically. The combination of these directions of interest has led to the development of refined techniques for the analysis and synthesis of feedback systems.

For example, the dynamic characteristics of feedback systems are of very basic importance for the communications, systems, and instrumentation engineers. The various aspects of the central problem of designing systems for specified dynamic performance are most nearly unified in the field of automatic control, and it is in this field that the greatest part of the research has been done. Only a few years ago, the feedback-control engineer was in many ways a parasite, adapting the methods of the mechanical engineer, the communication engineer, etc., to his problems. Today, as a result of the tremendous research effort of the past decade, the feedback-control engineer has not only brought the various phases of engineering together, but he has assumed at least his share of engineering leadership.

Indeed, in many respects, servomechanisms and automatic control systems today comprise one of the glamorous fields of electrical engineering. The importance and popularity of this field are in no small measure the result of military applications in the development of fire-control systems, missile-control systems, etc. But already the many peacetime applications of feedback control systems are becoming evident. This rapid expansion of feedback-control-system engineering has demonstrated the very fundamental importance of the dynamic characteristics of systems. In the vast majority of servomechanisms, the designer is interested not only in the sinusoidal response of the system, but also in the response to typical or test transient signals.

An aircraft-to-aircraft fire-control system can be used to illustrate the importance of the time-domain characterization of a system. The intelligent design of such a system demands consideration of the actual waveform of the anticipated input signals. A typical target run may last only a short length of time, perhaps 10 sec. During this time interval, the radar must lock on the target and thereafter follow within a specified accuracy. There are two distinct design problems: the realization of a system which locks on within a very few seconds, even if the decision to lock on is made when the antenna is a considerable distance from the selected target; and the maintenance of suitable tracking accuracy after lock-on, with the probable relative maneuverings of the two aircraft taken into consideration. If only these factors were involved, the design would resolve simply to the realization of system components with a response sufficiently fast to meet the specifications. Design is complicated, however, by the presence of noise corrupting the signal. In a broad sense, design must determine a compromise between speed of response and filtering. The logical selection of this compromise can be accomplished only in terms of the transient response of the system to typical input signals. Ultimately, if the specifications are sufficiently difficult to meet, design must yield a system which separates signals from noise, even though both components of the input have the same frequency spectrum. Such a separation might be accomplished by a nonlinear filter on the basis of the difference in the probability distributions of signal and noise.

This fire-control problem is particularly appropriate for an introductory discussion, because the design problems involved encompass a number of the important aspects of the modern theory of feedback control systems:

(1) The complexity of the complete system, including the tracking loop (radar, antenna servo, gyros to introduce corrections for own-ship's motion, etc.), the computer (for determining desired gun elevation and bearing from the estimated relative target motion, the ballistics, and the range), and the gun-positioning servos, indicates the need for a systematic approach to analysis and design. In the design of both the individual components and the over-all, multiloop system, the basic techniques of general network and feedback theory are essential tools.

(2) The problem emphasizes the importance of the transient response of the final system. Indeed, all possible input functions can be described by a class of transient signals. Evaluation of the tracking accuracy (even in the absence of noise) requires characterization of the system in the time domain.

(3) For a complete design, characteristics of the noise must be known, and the noise power must be determined at various points throughout the system before an evaluation of system performance can be completed. Since the noise is ordinarily random and can be described only in statistical terms, design requires some familiarity with statistical methods of measurement and analysis.

(4) The effects of the noise are most readily analyzed in the frequency domain, *i.e.*, in terms of the noise power spectrum (measuring the noise power in certain frequency bands) and the gain-frequency characteristics of the physical components.

(5) Considerations (2) and (4) indicate clearly that design is most effective if system components are described in such a way that both time-domain and frequency-domain characteristics are evident. The Laplace transform, with the concept of complex frequency $s = \sigma + j\omega$,

provides the required correlation between transient and frequency responses, but only if the transfer functions are considered as functions of s, not if the components are characterized by the gain and phase versus frequency.

(6) The fire-control problem is one in which the ultimate design must be for a nonlinear system, particularly if the system performance is improved to the point where significant errors are introduced by considering the noise characteristics to be constant (when in actuality these characteristics change rather markedly with range). Appropriate methods for the logical design of nonlinear systems are certainly not general at the present time, but, once the properties and limitations of the linear system are clearly understood, a variety of special procedures for analyzing and (in part) designing nonlinear systems is available.

Thus, one of the most important changes in communication and control engineering during the last decade has been the broadening of interest from the frequency characteristics to the performance characteristics with the system excited by transient inputs or by actual, typical inputs described statistically. Techniques for the design and synthesis of networks and feedback systems have been extended to admit control over both time-domain and frequency-domain characteristics. The increased interest in nonlinear systems has provided further impetus to the development of such time-domain characterization. Since superposition does not apply in a nonlinear system, it is no longer possible in the design to justify the use of frequency characteristics by stating that the Fourier integral permits any aperiodic signal to be represented as the sum of sine waves, and, accordingly, design cannot be carried through in terms of frequency characteristics.

The necessity for correlation between time-domain and frequency-domain characteristics has been met in the design of linear systems by exploiting the Laplace transform and the associated complex-function theory as a mathematical tool in synthesis. The characterization of a linear system by a transfer function depending on the complex frequency s permits the designer to consider simultaneously both transient and frequency characteristics. The elements of Laplace-transform theory and complex-function theory are presented in almost all texts on feedback control systems. In this introductory chapter, certain aspects of these theories are considered in detail, and aspects of particular interest in the analysis and synthesis procedures are presented in subsequent chapters.† To a very large extent, the power of the theory presented throughout this book depends on a familiarity with the introductory mathematics of this chapter.

† At the outset, a knowledge of the very elementary aspects of Laplace-transform theory is assumed. Appropriate discussions are presented in G. S. Brown and D. P. Campbell, "Principles of Servomechanisms," Chap. 3, John Wiley & Sons, Inc., New York, 1948; in W. R. Ahrendt and J. F. Taplin, "Automatic Feedback Control," Chap. 2, McGraw-Hill Book Company, Inc., New York, 1951; and in H. Chestnut and R. W. Mayer, "Servomechanisms and Regulating System Design," Vol. I, Chap. 4, John Wiley & Sons, Inc., New York, 1951.

1.1. Functions of a Complex Variable. If the transfer function of the network shown in Fig. 1.1 is written as a function of the complex frequency s,

$$\frac{E_2}{E_1}(s) = \frac{1/Cs}{R + 1/Cs} = \frac{1}{RCs + 1} \tag{1.1}$$

The value of the voltage ratio at any sinusoidal angular frequency ω is found by replacing s by $j\omega$; the gain and phase characteristics are then given by the magnitude and angle of $\frac{E_2}{E_1}(j\omega)$. Since $\frac{E_2}{E_1}(s)$ is also the ratio of the Laplace transforms of the output and input voltages, Eq. (1.1) contains not only the gain and phase characteristics, but also the transient response of the network.

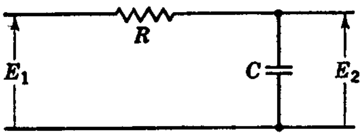

FIG. 1.1. Simple transfer network.

The analysis of more complicated networks follows the same pattern: by means of conventional techniques (loop equations, node equations, etc.), the transfer function is determined in terms of the complex frequency s. If the network is linear and consists of lumped-constant elements, the resulting transfer function, whether it be a voltage ratio, a current ratio, an impedance, or an admittance, is always a *rational algebraic* function of s (that is, the ratio of two polynomials in s). Indeed, this rational algebraic nature is realized regardless of the type of system: *i.e.*, whether electrical, mechanical, hydraulic, etc.

As a consequence of the use of the complex frequency $s = \sigma + j\omega$, analysis of the characteristics of transfer functions can draw upon the mathematicians' well-developed theory of functions of a complex variable. Indeed, the analysis and synthesis of networks and feedback systems are to a very large extent the application of complex-function theory. A few particularly useful aspects of this theory, described briefly in this section, form the basis for the network and feedback theory presented in subsequent chapters.†

Definition. The functions of a complex variable of principal interest are those which possess a unique derivative. The derivative of a real function of a single variable x is defined by the equation

$$f'(x) = \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right] \tag{1.2}$$

† Fortunately, a wide variety of excellent texts are available to present the material with more rigor and in considerably more detail. Complex-function theory is considered at least briefly in almost all texts on advanced mathematics for electrical engineers: *e.g.*, E. A. Guillemin, "Mathematics of Circuit Analysis," Chap. 6, John Wiley & Sons, Inc., New York, 1949; and A. Bronwell, "Advanced Mathematics in Physics and Engineering," Chap. 15, McGraw-Hill Book Company, Inc., New York, 1953. Mathematical texts on complex-function theory are numerous. Two that the author has particularly liked are K. Knopp, "Theory of Functions," Dover Publications, New York, 1945; and E. T. Copson, "Theory of Functions of a Complex Variable," Oxford University Press, London, 1935.

If this definition is extended to functions of the complex variable s,

$$G'(s) = \lim_{\Delta s \to 0} \left[\frac{G(s + \Delta s) - G(s)}{\Delta s} \right] \tag{1.3}$$

If the $G(s)$ is an arbitrary function of s, it is clear that the value of $G'(s)$ at any point s_1 may depend on the Δs used: for example, on the angle of Δs (*i.e.*, the direction from s_1 along which the increment is considered). If such a nonuniqueness exists, $G(s)$ is not ordinarily of interest.

A simple example illustrates the difficulty here. The $G(s)$ considered is

$$G(s) = |s| \tag{1.4}$$

or

$$G(s) = \sqrt{\sigma^2 + \omega^2} \tag{1.5}$$

If an attempt is made to evaluate the derivative $G'(s)$ by considering the change in the function as s changes by a real increment $\Delta\sigma$,

$$G'(s) = \lim_{\Delta\sigma \to 0} \left[\frac{\sqrt{(\sigma + \Delta\sigma)^2 + \omega^2} - \sqrt{\sigma^2 + \omega^2}}{\Delta\sigma} \right] \tag{1.6}$$

The numerator can be rewritten to give

$$G'(s) = \lim_{\Delta\sigma \to 0} \left[\sqrt{\sigma^2 + \omega^2}\, \frac{\sqrt{1 + \dfrac{2\sigma\,\Delta\sigma + (\Delta\sigma)^2}{\sigma^2 + \omega^2}} - 1}{\Delta\sigma} \right] \tag{1.7}$$

As $\Delta\sigma$ tends to zero, Eq. (1.7) becomes

$$G'(s) = \lim_{\Delta\sigma \to 0} \left[\sqrt{\sigma^2 + \omega^2}\, \frac{\sigma\,\Delta\sigma/(\sigma^2 + \omega^2)}{\Delta\sigma} \right] \tag{1.8}$$

$$G'(s) = \frac{\sigma}{\sqrt{\sigma^2 + \omega^2}} \tag{1.9}$$

If a Δs of $j\Delta\omega$ is used instead of $\Delta\sigma$, Eq. (1.3) gives the relation

$$G'(s) = \frac{-j\omega}{\sqrt{\sigma^2 + \omega^2}} \tag{1.10}$$

In general, Eqs. (1.9) and (1.10) indicate two entirely different values of $G'(s)$. This difficulty in defining a derivative limits severely the permissible mathematical manipulations on $G(s)$. Consequently, in order that a unified theory may be constructed, such $G(s)$ functions are not considered.

The usual functions encountered in network theory and control-system analysis possess a unique derivative at almost all points in the s plane. If $G(s)$ is a rational algebraic function, the derivative exists in the entire s

plane except at the isolated points representing the zeros of the denominator, where $G(s)$ and all derivatives become infinite.

Analyticity. $G(s)$ is analytic in a region if the function and all its derivatives exist in the region. For example, the function $G(s) = 1/(s+1)$ is analytic in any region of the s plane not including the point -1. The function $\sqrt{s}$ is analytic in any region not including the origin, since at $s = 0$ the first and all higher derivatives do not exist.

In exceptional cases in which doubt exists as to whether a given $G(s)$ is an analytic function of a complex variable, the Cauchy-Riemann equations provide a useful test. $G(s)$ is an analytic function of the complex variable s if

$$\begin{aligned} \frac{\partial(\operatorname{Re} G)}{\partial\sigma} &= \frac{\partial(\operatorname{Im} G)}{\sigma\omega} \\ \frac{\partial(\operatorname{Im} G)}{\partial\sigma} &= -\frac{\partial(\operatorname{Re} G)}{\partial\omega} \end{aligned} \tag{1.11}$$

and all four partial derivatives are continuous functions of s. Throughout the remainder of this section, all functions considered satisfy these conditions except at isolated points in the s plane.

Singularities. The points at which a function (or its derivatives) does not exist are termed the singularities of the function. In this section, the functions of interest are those with only isolated singularities in the finite part of the plane: *i.e.*, in any finite region there are only a finite number of singularities, or, more rigorously, the singularities do not possess a limit point in the finite part of the s plane.

The singularities are of fundamental importance in characterizing the function of a complex variable, for the location and nature of the singularities determine the behavior of the function throughout the entire plane. Because of this importance, the singularities are the basis for the classification of functions. For example, a function with no singularities in the s plane (including the point at infinity) must be a constant. A function with no finite singularities, but possibly with a singularity at infinity, is termed an *entire function.* Functions with finite singularities are divided according to the nature and number of singularities.

There are three types of singularities of importance in the analysis and design of feedback systems:

(1) Poles
(2) Essential singularities
(3) Branch points

The classification indicates the manner in which the function (or a derivative) behaves as s approaches the singularity, although ordinarily the various types are defined in analytical terms.

The simplest type of singularity, a pole, is defined as follows: if a positive *integer* n can be found such that

$$\lim_{s \to s_1} [(s - s_1)^n G(s)]$$

has a nonzero, finite value, s_1 is a pole of $G(s)$, and the pole is of order n.† In other words, the denominator of $G(s)$ must include, at least implicitly, the multiplicative factor $(s - s_1)^n$. Since any rational algebraic function can be written as the ratio of factored polynomials,

$$\frac{(s-a)(s-b)\cdots(s-m)}{(s-\alpha)(s-\beta)\cdots(s-\nu)}$$

all singularities of rational algebraic functions are poles. In the form above, the poles are at $\alpha, \beta, \ldots, \nu$. If j of the denominator factors are identical, one of the poles is of order j.

An essential singularity can be viewed as a pole of infinite order. Rigorously, an essential singularity should be defined in terms of the Laurent expansion.‡ For the applications here, however, the rigorous definition is not especially important, since most essential singularities encountered in network theory are associated with exponential functions. The function e^{-s}, for example, possesses an essential singularity at infinity, while $e^{-1/s}$ has an essential singularity at the origin.

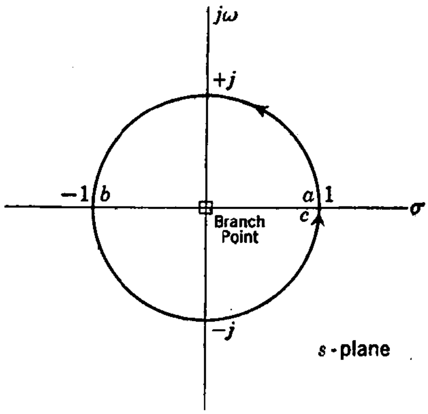

FIG. 1.2. Closed path around a branch point. $G(s) = \sqrt{s}$.

Branch points are singularities characterized by the phenomenon that complete traversal of a path enclosing a branch point does not result in a return to the original value of the function. $\sqrt{s}$ is the simplest function with a branch point, in this case at the origin. If the path shown in Fig. 1.2 is followed, the value of $\sqrt{s}$ starts as $\sqrt{e^{j0}} = e^{j0}$, or $+1$; when point b is reached, the value is $\sqrt{e^{j\pi}}$, or $+j$; as point c is approached, the value tends to $\sqrt{e^{j2\pi}}$, or -1, and the multivalued character of the function is apparent. A single encirclement of the branch point at the origin results in a change in the value of $G(s)$. In this case, a second encirclement would result in the original value of the function; consequently, the branch point is of order two. The logarithmic function $\ln s$ is an example of a function with a branch point of infinite order, since every positive encirclement of the origin increases the imaginary part of $\ln s$ by 2π. Branch points also may be characterized by the behavior of $G(s)$ as s approaches the singularity. Thus, $\sqrt{s}$ has a zero-type branch point at

† A pole is an isolated singularity. In other words, if the pole is at s_1, it is always possible to find a small circular region around s_1 such that no other singularities lie within this region.

‡ The Laurent expansion about an essential singularity contains an infinite number of terms. As ordinarily defined, an essential singularity need not be an isolated singularity but may, for example, be the limit point of a sequence of poles.