A. A. Sveshnikov

PROBLEMS IN PROBABILITY THEORY, MATHEMATICAL STATISTICS AND THEORY OF RANDOM FUNCTIONS

Problems in Probability Theory, Mathematical Statistics and Theory of Random Functions

Edited by A. A. SVESHNIKOV

Translated by Scripta Technica, Inc.
Edited by Bernard R. Gelbaum

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Foreword

Students at all levels of study in the theory of probability and in the theory of statistics will find in this book a broad and deep cross-section of problems (and their solutions) ranging from the simplest combinatorial probability problems in finite sample spaces through information theory, limit theorems and the use of moments.

The introductions to the sections in each chapter establish the basic formulas and notation and give a general sketch of that part of the theory that is to be covered by the problems to follow. Preceding each group of problems, there are typical examples and their solutions carried out in great detail. Each of these is keyed to the problems themselves so that a student seeking guidance in the solution of a problem can, by checking through the examples, discover the appropriate technique required for the solution.

Bernard R. Gelbaum

Contents

I RANDOM EVENTS

1. RELATIONS AMONG RANDOM EVENTS

Basic Formulas

Random events are usually designated by the letters $A, B, C, \ldots, U, V$, where U denotes an event certain to occur and V an impossible event. The equality $A = B$ means that the occurrence of one of the events inevitably brings about the occurrence of the other. The intersection of two events A and B is defined as the event $C = AB$, said to occur if and only if both events A and B occur. The union of two events A and B is the event $C = A \cup B$, said to occur if and only if at least one of the events A and B occurs. The difference of two events A and B is defined as the event $C = A \setminus B$, said to occur if and only if A occurs and B does not occur. The complementary event is denoted by the same letter as the initial event, but with an over bar. For instance, $\bar{A}$ and A are complementary, $\bar{A}$ meaning that A does not occur. Two events are said to be mutually exclusive if $AB = V$. The events A_k ($k = 1, 2, \ldots, n$) are said to form a complete set if the experiment results in at least one of these events so that $\bigcup_{k=1}^{n} A_k = U$.

Solution for Typical Examples

Example 1.1 What kind of events A and B will satisfy the equality $A \cup B = A$?

Solution. The union $A \cup B$ means the occurrence of at least one of the events A and B. Then, for $A \cup B = A$, the event A must include the event B. For example, if A means falling into region S_A and B falling into region S_B, then S_B lies within S_A.

The solution to Problems 1.1 to 1.3 and 1.8 is similar.

Example 1.2 Two numbers at random are selected from a table of random numbers. If the event A means that at least one of these numbers is prime and the event B that at least one of them is an even number, what is the meaning of events AB and $A \cup B$?

Solution. Event AB means that both events A and B occur. The event $A \cup B$ means that at least one of the two events occurs; that is, from two selected numbers at least one number is prime or one is even, or one number is prime and the other is even.

One can solve Problems 1.4 to 1.7 analogously.

Example 1.3 Prove that $\overline{\bar{A}\bar{B}} = A \cup B$ and $\overline{\bar{C} \cup \bar{D}} = CD$.

PROOF. If $C = \bar{A}$ and $D = \bar{B}$, the second equality can be written in the form $\overline{A \cup B} = \bar{A}\bar{B}$. Hence it suffices to prove the validity of the first equality.

The event $\bar{A}\bar{B}$ means that both events A and B do not occur. The complementary event $\overline{\bar{A}\bar{B}}$ means that at least one of these events occurs: the union $A \cup B$. Thus $\overline{\bar{A}\bar{B}} = A \cup B$. The proof of this equality can also be carried out geometrically, an event meaning that a point falls into a certain region.

One can solve Problem 1.9 similarly. The equalities proved in Example 1.3 are used in solving Problems 1.10 to 1.14.

Example 1.4 The scheme of an electric circuit between points M and N is represented in Figure 1. Let the event A be that the element a is out of order, and let the events B_k ($k = 1, 2, 3$) be that an element b_k is out of order. Write the expressions for C and $\bar{C}$, where the event C means the circuit is broken between M and N.

SOLUTION. The circuit is broken between M and N if the element a or the three elements b_k ($k = 1, 2, 3$) are out of order. The corresponding events are A and $B_1B_2B_3$. Hence $C = A \cup B_1B_2B_3$.

Using the equalities of Example 1.3, we find that

$$\bar{C} = \overline{A \cup B_1B_2B_3} = \bar{A}\overline{B_1B_2B_3} = \bar{A}(\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3).$$

Similarly one can solve Problems 1.16 to 1.18.

PROBLEMS

1.1 What meaning can be assigned to the events $A \cup A$ and AA?

1.2 When does the equality $AB = A$ hold?

1.3 A target consists of 10 concentric circles of radius r_k ($k = 1, 2, 3, \ldots, 10$). An event A_k means hitting the interior of a circle of radius r_k ($k = 1, 2, \ldots, 10$). What do the following events mean:

$$B = \bigcup_{k=1}^{6} A_k, \qquad C = \prod_{k=5}^{10} A_k\,?$$

1.4 Consider the following events: A that at least one of three devices checked is defective, and B that all devices are good. What is the meaning of the events (a) $A \cup B$, (b) AB?

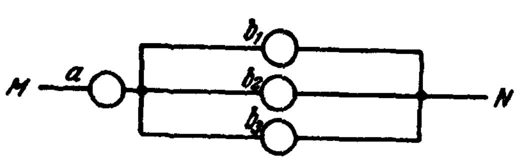


FIGURE 1

1.5 The events A, B and C mean selecting at least one book from three different collections of complete works; each collection consists of at least three volumes. The events A_s and B_k mean that s volumes are taken from the first collection and k volumes from the second collection. Find the meaning of the events (a) $A \cup B \cup C$, (b) ABC, (c) $A_1 \cup B_3$, (d) A_2B_2, (e) $(A_1B_3 \cup B_1A_3)C$.

1.6 A number is selected at random from a table of random numbers. Let the event A be that the chosen number is divisible by 5, and let the event B be that the chosen number ends with a zero. Find the meaning of the events $A \setminus B$ and $\overline{AB}$.

1.7 Let the event A be that at least one out of four items is defective, and let the event B be that at least two of them are defective. Find the complementary events $\bar{A}$ and $\bar{B}$.

1.8 Simplify the expression $A = (B \cup C)(B \cup \bar{C})(\bar{B} \cup C)$.

1.9 When do the following equalities hold true: (a) $A \cup B = \bar{A}$, (b) $AB = \bar{A}$, (c) $A \cup B = AB$?

1.10 From the following equality find the random event X:

$$\overline{\overline{X} \cup A} \cup \overline{\overline{X} \cup \bar{A}} = B.$$

1.11 Prove that $\bar{A}B \cup A\bar{B} = \overline{AB}$.

1.12 Prove that the following two equalities are equivalent:

$$\overline{\bigcup_{k=1}^{n} A_k} = \prod_{k=1}^{n} \bar{A}_k, \qquad \bigcup_{k=1}^{n} \bar{A}_k = \overline{\prod_{k=1}^{n} A_k}.$$

1.13 Can the events A and $\overline{A \cup B}$ be simultaneous?

1.14 Prove that A, $\bar{A}B$ and $\overline{A \cup B}$ form a complete set of events.

1.15 Two chess players play one game. Let the event A be that the first player wins, and let B be that the second player wins. What event should be added to these events to obtain a complete set?

1.16 An installation consists of two boilers and one engine. Let the event A be that the engine is in good condition, let B_k $(k = 1, 2)$ be that the kth boiler is in good condition, and let C be that the installation can operate if the engine and at least one of the boilers are in good condition. Express the events C and $\bar{C}$ in terms of A and B_k.

1.17 A vessel has a steering gear, four boilers and two turbines. Let the event A be that the steering gear is in good condition, let B_k $(k = 1, 2, 3, 4)$ be that the boiler labeled k is in good condition, let C_j $(j = 1, 2)$ be that the turbine labeled j is in good condition, and let D be that the vessel can sail if the engine, at least one of the boilers and at least one of the turbines are in good condition. Express D and $\bar{D}$ in terms of A and B_k.

1.18 A device is made of two units of the first type and three units of the second type. Let A_k $(k = 1, 2)$ be that the kth unit of the first type is in good condition, let B_j $(j = 1, 2, 3)$ be that the jth unit of the second type is in good condition, and let C be that the device can operate if at least one unit of the first type and at least two units of the second type are in good condition. Express the event C in terms of A_k and B_j.

2. A DIRECT METHOD FOR EVALUATING PROBABILITIES

Basic Formulas

If the outcomes of an experiment form a finite set of n elements, we shall say that the outcomes are equally probable if the probability of each outcome is $1/n$. Thus if an event consists of m outcomes, the probability of the event is $p = m/n$.

SOLUTION FOR TYPICAL EXAMPLES

Example 2.1 A cube whose faces are colored is split into 1000 small cubes of equal size. The cubes thus obtained are mixed thoroughly. Find the probability that a cube drawn at random will have two colored faces.

SOLUTION. The total number of small cubes is $n = 1000$. A cube has 12 edges so that there are eight small cubes with two colored faces on each edge. Hence $m = 12 \cdot 8 = 96$, $p = m/n = 0.096$.

Similarly one can solve Problems 2.1 to 2.7.

Example 2.2 Find the probability that the last two digits of the cube of a random integer will be 1.[1]

SOLUTION. Represent N in the form $N = a + 10b + \cdots$, where $a, b, \ldots$ are arbitrary numbers ranging from 0 to 9. Then $N^3 = a^3 + 30a^2b + \cdots$. From this we see that the last two digits of N^3 are affected only by the values of a and b. Therefore the number of possible values is $n = 100$. Since the last digit of N^3 is a 1, there is one favorable value $a = 1$. Moreover, the last digit of $(N^3 - 1)/10$ should be 1; i.e., the product $3b$ must end with a 1. This occurs only if $b = 7$. Thus the favorable value ($a = 1$, $b = 7$) is unique and, therefore, $p = 0.01$.

Similarly one can solve Problems 2.8 to 2.11.

Example 2.3 From a lot of n items, k are defective. Find the probability that l items out of a random sample of size m selected for inspection are defective.

SOLUTION. The number of possible ways to choose m items out of n is C_n^m. The favorable cases are those in which l defective items among the k defective items are selected (this can be done in C_k^l ways), and the remaining $m - l$ items are nondefective, i.e., they are chosen from the total number $n - k$ (in C_{n-k}^{m-l} ways). Thus the number of favorable cases is $C_k^l C_{n-k}^{m-l}$. The required probability will be $p = (C_k^l C_{n-k}^{m-l})/C_n^m$.

One can solve Problems 2.12 to 2.20 similarly.

Example 2.4 Five pieces are drawn from a complete domino set. Find the probability that at least one of them will have six dots marked on it.

SOLUTION. Find the probability q of the complementary event. Then $p = 1 - q$. The probability that all five pieces will not have a six (see Example 2.3) is $q = (C_7^0 C_{21}^5)/C_{28}^5$ and, hence,

$$p = 1 - \frac{C_{21}^5}{C_{28}^5} = 0.793.$$

[1] By a "random number" here we mean a k-digit number ($k > 1$) such that any of its digits may range from 0 to 9 with equal probability.

By a similar passage to the complementary event, one can solve Problems 2.21 and 2.22.

PROBLEMS

2.1 Lottery tickets for a total of n dollars are on sale. The cost of one ticket is r dollars, and m of all tickets carry valuable prizes. Find the probability that a single ticket will win a valuable prize.

2.2 A domino piece selected at random is not a double. Find the probability that the second piece also selected at random, will match the first.

2.3 There are four suits in a deck containing 36 cards. One card is drawn from the deck and returned to it. The deck is then shuffled thoroughly and another card is drawn. Find the probability that both cards drawn belong to the same suit.

2.4 A letter combination lock contains five disks on a common axis. Each disk is divided into six sectors with different letters on each sector. The lock can open only if each of the disks occupies a certain position with respect to the body of the lock. Find the probability that the lock will open for an arbitrary combination of the letters.

2.5 The black and white kings are on the first and third rows, respectively, of a chess board. The queen is placed at random in one of the free squares of the first or second row. Find the probability that the position for the black king becomes checkmate if the positions of the kings are equally probable in any squares of the indicated rows.

2.6 A wallet contains three quarters and seven dimes. One coin is drawn from the wallet and then a second coin, which happens to be a quarter. Find the probability that the first coin drawn is a quarter.

2.7 From a lot containing m defective items and n good ones, s items are chosen at random to be checked for quality. As a result of this inspection, one finds that the first k of s items are good. Determine the probability that the next item will be good.

2.8 Determine the probability that a randomly selected integer N gives as a result of (a) squaring, (b) raising to the fourth power, (c) multiplying by an arbitrary integer, a number ending with a 1.

2.9 On 10 identical cards are written different numbers from 0 to 9. Determine the probability that (a) a two-digit number formed at random with the given cards will be divisible by 18, (b) a random three-digit number will be divisible by 36.

2.10 Find the probability that the serial number of a randomly chosen bond contains no identical digits if the serial number may be any five-digit number starting with 00001.

2.11 Ten books are placed at random on one shelf. Find the probability that three given books will be placed one next to the other.

2.12 The numbers 2, 4, 6, 7, 8, 11, 12 and 13 are written, respectively, on eight indistinguishable cards. Two cards are selected at random from the eight. Find the probability that the fraction formed with these two random numbers is reducible.

2.13 Given five segments of lengths 1, 3, 5, 7 and 9 units, find the probability that three randomly selected segments of the five will be the sides of a triangle.

2.14 Two of 10 tickets are prizewinners. Find the probability that among five tickets taken at random (a) one is a prizewinner, (b) two are prizewinners, (c) at least one is a prizewinner.

2.15 This is a generalization of Problem 2.14. There are $n + m$ tickets of which n are prizewinners. Someone purchases k tickets at the same time. Find the probability that s of these tickets are winners.

2.16 In a lottery there are 90 numbers, of which five win. By agreement one can bet any sum on any one of the 90 numbers or any set of two, three, four or five numbers. What is the probability of winning in each of the indicated cases?

2.17 To decrease the total number of games, $2n$ teams have been divided into two subgroups. Find the probability that the two strongest teams will be (a) in different subgroups, (b) in the same subgroup.

2.18 A number of n persons are seated in an auditorium that can accommodate $n + k$ people. Find the probability that $m \leqslant n$ given seats are occupied.

2.19 Three cards are drawn at random from a deck of 52 cards. Find the probability that these three cards are a three, a seven and an ace.

2.20 Three cards are drawn at random from a deck of 36 cards. Find the probability that the sum of points of these cards is 21 if the jack counts as two points, the queen as three points, the king as four points, the ace as eleven points and the rest as six, seven, eight, nine and ten points.

2.21 Three tickets are selected at random from among five tickets worth one dollar each, three tickets worth three dollars each and two tickets worth five dollars each. Find the probability that (a) at least two of them have the same price, (b) all three of them cost seven dollars.

2.22 There are $2n$ children in line near a box office where tickets priced at a nickel each are sold. What is the probability that nobody will have to wait for change if, before a ticket is sold to the first customer, the cashier has $2m$ nickels and it is equally probable that the payments for each ticket are made by a nickel or by a dime.

3. GEOMETRIC PROBABILITIES

Basic Formulas

The geometric definition of probability can be used only if the probability of hitting any part of a certain domain is proportional to the size of this domain (length, area, volume, and so forth), and is independent of its position and shape.

If the geometric size of the whole domain equals S, the geometric size of a part of it equals S_B, and a favorable event means hitting S_B, then the probability of this event is defined to be

$$p = \frac{S_B}{S}.$$

The domains can have any number of dimensions.

SOLUTION FOR TYPICAL EXAMPLES

Example 3.1 The axes of indistinguishable vertical cylinders of radius r pass through an interval l of a straight line AB, which lies in a horizontal plane. A ball of radius R is thrown at an angle q to this line. Find the probability that this ball will hit one cylinder if any intersection point of the path described by the center of the ball with the line AB is equally probable.[2]

SOLUTION. Let x be the distance from the center of the ball to the nearest line that passes through the center of a cylinder parallel to the displacement direction of the center of the ball. The possible values of x are determined by the conditions (Figure 2)

$$0 \leqslant x \leqslant \frac{1}{2} l \sin q.$$

The collision of the ball with the cylinder may occur only if $0 \leqslant x \leqslant R + r$.

The required probability equals the ratio between the length of the segment on which lie the favorable values of x and the length of the segment on which lie all the values of x. Consequently,

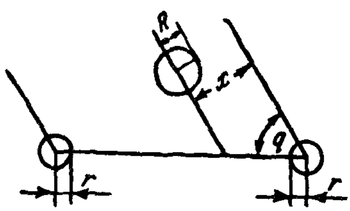
$$p = \begin{cases} \dfrac{2(R+r)}{l \sin q} & \text{for } R + r \leqslant \dfrac{l}{2} \sin q; \\ 1 & \text{for } R + r \geqslant \dfrac{l}{2} \sin q. \end{cases}$$

One can solve problems 3.1 to 3.4 and 3.24 analogously.

Example 3.2 On one track of a magnetic tape 200 m. long some information is recorded on an interval of length 20 m., and on the second track similar information is recorded. Estimate the probability that from 60 to 85 m. there is no interval on the tape without recording if the origins of both recordings are located with equal probability at any point from 0 to 180 m.

SOLUTION. Let x and y be the coordinates of origin of the recordings, where $x \geqslant y$. Since $0 \leqslant x \leqslant 180$, $0 \leqslant y \leqslant 180$ and $x \geqslant y$, the domain of all the possible values of x and y is a right triangle with hypotenuse 180 m. The area of this triangle is $S = 1/2 \cdot 180^2$ sq. m. Find the domain of values of x and y

FIGURE 2



[2] The restriction of equal probability used in formulating several problems with a point that hits the interior of any part of a domain (linear, two-dimensional, and so forth) is understood only in connection with the notion of *geometric probability*.

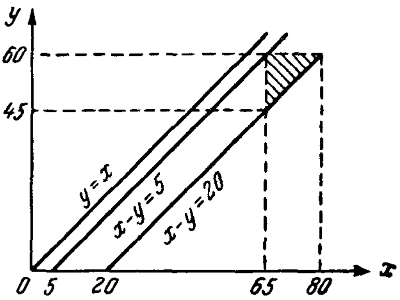


FIGURE 3

favorable to the given event. To obtain a continuous recording, it is necessary that the inequality $x - y \leqslant 20$ m. hold true. To obtain a recording interval longer than or equal to 25 m., we must have $x - y \geqslant 5$ m. Moreover, to obtain a continuous recording on the interval from 60 to 85 m., we must have

$$45 \text{ m.} \leqslant y \leqslant 60 \text{ m.},$$

$$65 \text{ m.} \leqslant x \leqslant 80 \text{ m}.$$

Drawing the boundaries of the indicated domains, we find that the favorable values of x and y are included in a triangle whose area $S_B = 1/2 \cdot 15^2$ sq. m. (Figure 3). The required probability equals the ratio of the area S_B favorable to the given event and the area of the domain S containing all possible values of x and y, namely,

$$p = \left(\frac{15}{180}\right)^2 = \frac{1}{144}.$$

One can solve Problems 3.5 to 3.15 similarly.

Example 3.3 It is equally probable that two signals reach a receiver at any instant of the time T. The receiver will be jammed if the time difference in the reception of the two signals is less than τ. Find the probability that the receiver will be jammed.

SOLUTION. Let x and y be the instants when the two signals are received. The domain of all the possible values of x, y is a square of area T^2 (Figure 4).

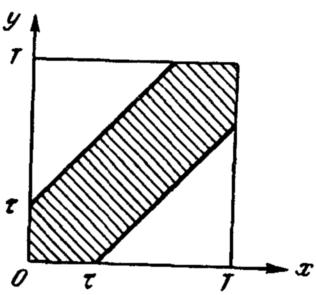


FIGURE 4

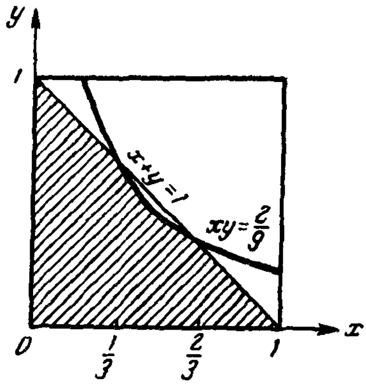


FIGURE 5

The receiver will be jammed if $|x - y| \leqslant \tau$. The given domain lies between the straight lines $x - y = \tau$ and $x - y = -\tau$. Its area equals

$$S_B = S - (T - \tau)^2,$$

and, therefore,

$$p = 1 - \left(1 - \frac{\tau}{T}\right)^2.$$

One can solve Problems 3.16 to 3.19 analogously.

Example 3.4 Find the probability that the sum of two random positive numbers, each of which does not exceed one, will not exceed one, and that their product will be at most 2/9.

SOLUTION. Let x and y be the chosen numbers. Their possible values are $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, defining in the plane a square of area $S = 1$. The favorable values satisfy the conditions $x + y \leqslant 1$ and $xy \leqslant 2/9$. The boundary $x + y = 1$ divides the square in two so that the domain $x + y \leqslant 1$ represents the lower triangle (Figure 5). The second boundary $xy = 2/9$ is a hyperbola. The x's of the intersection points of these boundaries are: $x_1 = 1/3$ and $x_2 = 2/3$. The area of the favorable domain is

$$S_B = \frac{1}{3} + \int_{1/3}^{2/3} y\,dx = \frac{1}{3} + \frac{2}{9}\int_{1/3}^{2/3} \frac{dx}{x} = \frac{1}{3} + \frac{2}{9}\ln 2.$$

The desired probability is $p = S_B/S = 0.487$.

One can solve Problems 3.20 to 3.23 in a similar manner.

PROBLEMS

3.1 A break occurs at a random point on a telephone line AB of length L. Find the probability that the point C is at a distance not less than l from the point A.

3.2 Parallel lines are drawn in a plane at alternating distances of 1.5 and 8 cm. Estimate the probability that a circle of radius 2.5 cm. thrown at random on this plane will not intersect any line.

3.3 In a circle of radius R chords are drawn parallel to a given direction. What is the probability that the length of a chord selected at random will not exceed R if any positions of the intersection points of the chord with the diameter perpendicular to the given direction are equally probable?

3.4 In front of a disk rotating with a constant velocity we place a segment of length $2h$ in the plane of the disk so that the line joining the midpoint of the segment with the center of the disk is perpendicular to this segment. At an arbitrary instant a particle flies off the disk. Estimate the probability that the particle will hit the segment if the distance between the segment and the center of the disk is l.

3.5 A rectangular grid is made of cylindrical twigs of radius r. The distances between the axes of the twigs are a and b respectively. Find the probability that a ball of diameter d, thrown without aiming, will hit the grid in one trial if the flight trajectory of the ball is perpendicular to the plane of the grid.

3.6 A rectangle 3 cm. × 5 cm. is inscribed in an ellipse with the semi-axes $a = 100$ cm. and $b = 10$ cm. so that its larger side is parallel to a. Furthermore, one constructs four circles of diameter 4.3 cm. that do not intersect the ellipse, the rectangle and each other.

Determine the probability that (a) a random point whose position is equally probable inside the ellipse will turn out to be inside one of the circles, (b) the circle of radius 5 cm. constructed with the center at this point will intersect at least one side of the rectangle.

3.7 Sketch five concentric circles of radius kr, where $k = 1, 2, 3, 4, 5$, respectively. Shade the circle of radius r and two annuli with the corresponding exterior radii of $3r$ and $5r$. Then select at random a point in the circle of radius $5r$. Find the probability that this point will be in (a) the circle of radius $2r$, (b) the shaded region.

3.8 A boat, which carries freight from one shore of a bay to the other, crosses the bay in one hour. What is the probability that a ship moving along the bay will be noticed if the ship can be seen from the boat at least 20 minutes before the ship intersects the direction of the boat and at most 20 minutes after the ship intersects the direction of the boat? All times and places for intersection are equally likely.

3.9 Two points are chosen at random on a segment of length l. Find the probability that the distance between the points is less than kl, if $0 < k < l$.

3.10 Two points L and M are placed at random on a segment AB of length l. Find the probability that the point L is closer to M than to A.

3.11 On a segment of length l, two points are placed at random so that the segment is divided into three parts. Find the probability that these three parts of the segment are sides of a triangle.

3.12 Three points A, B, C are placed at random on a circle of radius R. What is the probability that the triangle ABC is acute-angled?

3.13 Three line segments, each of a length not exceeding l, are chosen at random. What is the probability that they can be used to form the sides of a triangle?