Real and Complex Analysis

SECOND EDITION

WALTER RUDIN

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**REAL AND
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PREFACE

This book contains a first-year graduate course in which the basic techniques and theorems of analysis are presented in such a way that the intimate connections between its various branches are strongly emphasized. The traditionally separate subjects of "real analysis" and "complex analysis" are thus united; some of the basic ideas from functional analysis are also included.

Here are some examples of the way in which these connections are demonstrated and exploited. The Riesz representation theorem and the Hahn-Banach theorem allow one to "guess" the Poisson integral formula. They team up in the proof of Runge's theorem. They combine with Blaschke's theorem on the zeros of bounded holomorphic functions to give a proof of the Müntz-Szasz theorem, which concerns approximation on an interval. The fact that L^2 is a Hilbert space is used in the proof of the Radon-Nikodym theorem, which leads to the theorem about differentiation of indefinite integrals (incidentally, differentiation seems to be unduly slighted in most modern texts), which in turn yields the existence of radial limits of bounded harmonic functions. The theorems of Plancherel and Cauchy combined give a theorem of Paley and Wiener which, in turn, is used in the Denjoy-Carleman theorem about infinitely differentiable functions on the real line. The maximum modulus theorem gives information about linear transformations on L^p-spaces.

Since most of the results presented here are quite classical (the novelty lies in the arrangement, and some of the proofs are new), I have not attempted to document the source of every item. References are gathered at the end, in Notes and Comments. They are not always to the original sources, but more often to more recent works where further references can be found. In no case does the absence of a reference imply any claim to originality on my part.

The prerequisite for this book is a good course in advanced calculus (set-theoretic manipulations, metric spaces, uniform continuity, and uniform convergence). The first seven chapters of my earlier book "Principles of Mathematical Analysis" furnish sufficient preparation.

Experience with the first edition shows that first-year graduate students can study the first 15 chapters in two semesters, plus some topics from 1 or 2 of the remaining 5. These latter are quite independent of each other. The first 15 should be taken up in the order in which they are presented, except for Chapter 9, which can be postponed.

Some new exercises have been added in this second edition, and many of the old ones have been regrouped so that they now appear in more or less the same order in which the corresponding topics occur in the text.

The text contains two substantial changes. The first of these was suggested by Jim Serrin, who showed me how to modify my earlier treatment of the differentiation of measures so as to obtain stronger results with no extra effort.

The second one is the inclusion of John Dixon's recently discovered, beautifully simple proof of the global (homology) version of Cauchy's theorem. This can now be proved and used as soon as some basic local properties of holomorphic functions are known. The order of several topics has accordingly been changed.

I have also made many smaller changes in order to improve some details and clarify some obscure points. Almost all of these were suggested by students, colleagues, and other friends. Their constructive comments and criticisms were greatly appreciated. I take this opportunity to thank them.

WALTER RUDIN

CONTENTS

PROLOGUE
THE EXPONENTIAL FUNCTION

This is the most important function in mathematics. It is defined, for every complex number z, by the formula

$$(1) \qquad \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The series (1) converges absolutely for every z and converges uniformly on every bounded subset of the complex plane. Thus exp is a continuous function. The absolute convergence of (1) shows that the computation

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{m=0}^{\infty} \frac{b^m}{m!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k} = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!}$$

is correct. It gives the important addition formula

$$(2) \qquad \exp(a)\exp(b) = \exp(a+b),$$

valid for all complex numbers a and b.

We define the number e to be $\exp(1)$, and shall usually replace $\exp(z)$ by the customary shorter expression e^z. Note that $e^0 = \exp(0) = 1$, by (1).

Theorem

(*a*) *For every complex* z *we have* $e^z \neq 0$.
(*b*) exp *is its own derivative*: $\exp'(z) = \exp(z)$.
(*c*) *The restriction of* exp *to the real axis is a monotonically increasing positive function, and*

$$e^x \to \infty \text{ as } x \to \infty, \qquad e^x \to 0 \text{ as } x \to -\infty.$$

(*d*) *There exists a positive number* π *such that* $e^{\pi i/2} = i$ *and such that* $e^z = 1$ *if and only if* $z/(2\pi i)$ *is an integer.*
(*e*) exp *is a periodic function, with period* $2\pi i$.
(*f*) *The mapping* $t \to e^{it}$ *maps the real axis onto the unit circle.*
(*g*) *If* w *is a complex number and* $w \neq 0$, *then* $w = e^z$ *for some* z.

PROOF By (2), $e^z \cdot e^{-z} = e^{z-z} = e^0 = 1$. This implies (*a*). Next,

$$\exp'(z) = \lim_{h \to 0} \frac{\exp(z+h) - \exp(z)}{h} = \exp(z) \lim_{h \to 0} \frac{\exp(h) - 1}{h} = \exp(z).$$

The first of the above equalities is a matter of definition, the second follows from (2), and the third from (1), and (*b*) is proved.

That exp is monotonically increasing on the positive real axis, and that $e^x \to \infty$ as $x \to \infty$, is clear from (1). The other assertions of (*c*) are consequences of $e^x \cdot e^{-x} = 1$.

For any real number t, (1) shows that e^{-it} is the complex conjugate of e^{it}. Thus

$$|e^{it}|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it} = e^{it-it} = e^0 = 1,$$

or

$$(3) \qquad |e^{it}| = 1 \qquad (t \text{ real}).$$

In other words, if t is real, e^{it} lies on the unit circle. We define $\cos t$, $\sin t$ to be the real and imaginary parts of e^{it}:

$$(4) \qquad \cos t = \operatorname{Re}[e^{it}], \qquad \sin t = \operatorname{Im}[e^{it}] \qquad (t \text{ real}).$$

If we differentiate both sides of Euler's identity

$$(5) \qquad e^{it} = \cos t + i \sin t,$$

which is equivalent to (4), and if we apply (*b*), we obtain

$$\cos' t + i \sin' t = ie^{it} = -\sin t + i \cos t,$$

so that

$$\cos' = -\sin, \qquad \sin' = \cos . \tag{6}$$

The power series (1) yields the representation

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots . \tag{7}$$

Take $t = 2$. The terms of the series (7) then decrease in absolute value (except for the first one) and their signs alternate. Hence $\cos 2$ is less than the sum of the first three terms of (7), with $t = 2$; thus $\cos 2 < -\frac{1}{3}$. Since $\cos 0 = 1$ and cos is a continuous real function on the real axis, we conclude that there is a smallest positive number t_0 for which $\cos t_0 = 0$. We define

$$\pi = 2t_0 . \tag{8}$$

It follows from (3) and (5) that $\sin t_0 = \pm 1$. Since

$$\sin'(t) = \cos t > 0$$

on the segment $(0, t_0)$ and since $\sin 0 = 0$, we have $\sin t_0 > 0$, hence $\sin t_0 = 1$, and therefore

$$e^{\pi i/2} = i. \tag{9}$$

It follows that $e^{\pi i} = i^2 = -1$, $e^{2\pi i} = (-1)^2 = 1$, and then $e^{2\pi i n} = 1$ for every integer n. Also, (*e*) follows immediately:

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z . \tag{10}$$

If $z = x + iy$, x and y real, then $e^z = e^x e^{iy}$; hence $|e^z| = e^x$. If $e^z = 1$, we therefore must have $e^x = 1$, so that $x = 0$; to prove that $y/2\pi$ must be an integer, it is enough to show that $e^{iy} \neq 1$ if $0 < y < 2\pi$, by (10).

Suppose $0 < y < 2\pi$, and

$$e^{iy/4} = u + iv \qquad (u \text{ and } v \text{ real}). \tag{11}$$

Since $0 < y/4 < \pi/2$, we have $u > 0$ and $v > 0$. Also

$$e^{iy} = (u + iv)^4 = u^4 - 6u^2v^2 + v^4 + 4iuv(u^2 - v^2). \tag{12}$$

The right side of (12) is real only if $u^2 = v^2$; since $u^2 + v^2 = 1$, this happens only when $u^2 = v^2 = \frac{1}{2}$, and then (12) shows that

$$e^{iy} = -1 \neq 1.$$

This completes the proof of (*d*).

We already know that $t \to e^{it}$ maps the real axis *into* the unit circle. To prove (*f*), fix w so that $|w| = 1$; we shall show that $w = e^{it}$ for some real t. Write $w = u + iv$, u and v real, and suppose first that $u \geq 0$ and $v \geq 0$. Since $u \leq 1$, the definition of π shows that there exists a t, $0 \leq t \leq \pi/2$, such that $\cos t = u$; then $\sin^2 t = 1 - u^2 = v^2$, and since $\sin t \geq 0$ if $0 \leq t \leq \pi/2$, we have $\sin t = v$. Thus $w = e^{it}$.

If $u < 0$ and $v \geq 0$, the preceding conditions are satisfied by $-iw$. Hence $-iw = e^{it}$ for some real t, and $w = e^{i(t+\pi/2)}$. Finally, if $v < 0$, the preceding two cases show that $-w = e^{it}$ for some real t, hence $w = e^{i(t+\pi)}$. This completes the proof of (*f*).

If $w \neq 0$, put $\alpha = w/|w|$. Then $w = |w|\alpha$. By (*c*), there is a real x such that $|w| = e^x$. Since $|\alpha| = 1$, (*f*) shows that $\alpha = e^{iy}$ for some real y. Hence $w = e^{x+iy}$. This proves (*g*) and completes the theorem. ////

We shall encounter the integral of $(1 + x^2)^{-1}$ over the real line. To evaluate it, put $\varphi(t) = \sin t/\cos t$ in $(-\pi/2, \pi/2)$. By (6), $\varphi' = 1 + \varphi^2$. Hence φ is a monotonically increasing mapping of $(-\pi/2, \pi/2)$ onto $(-\infty, \infty)$, and we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\pi/2}^{\pi/2} \frac{\varphi'(t)\,dt}{1 + \varphi^2(t)} = \int_{-\pi/2}^{\pi/2} dt = \pi.$$

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ABSTRACT INTEGRATION

Toward the end of the nineteenth century it became clear to many mathematicians that the Riemann integral (about which one learns in calculus courses) should be replaced by some other type of integral, more general and more flexible, better suited for dealing with limit processes. Among the attempts made in this direction, the most notable ones were due to Jordan, Borel, W. H. Young, and Lebesgue. It was Lebesgue's construction which turned out to be the most successful.

In brief outline, here is the main idea: The Riemann integral of a function f over an interval $[a, b]$ can be approximated by sums of the form

$$\sum_{i=1}^{n} f(t_i)m(E_i)$$

where $E_1, \ldots, E_n$ are disjoint intervals whose union is $[a, b]$, $m(E_i)$ denotes the length of E_i, and $t_i \in E_i$ for $i = 1, \ldots, n$. Lebesgue discovered that a completely satisfactory theory of integration results if the sets E_i in the above sum are allowed to belong to a larger class of subsets of the line, the so-called "measurable sets," and if the class of functions under consideration is enlarged to what he called "measurable functions." The crucial set-theoretic properties involved are the following: The union and the intersection of any countable family of measurable sets are measurable; so is the complement of every measurable set; and, most important, the notion of "length" (now called "measure") can be extended to them in such a way that

$$m(E_1 \cup E_2 \cup E_3 \cup \cdots) = m(E_1) + m(E_2) + m(E_3) + \cdots$$

for every countable collection $\{E_i\}$ of pairwise disjoint measurable sets. This property of m is called *countable additivity*.

The passage from Riemann's theory of integration to that of Lebesgue is a process of completion (in a sense which will appear more precisely later). It is of the same fundamental importance in analysis as is the construction of the real number system from the rationals.

The above-mentioned measure m is of course intimately related to the geometry of the real line. In this chapter we shall present an abstract (axiomatic) version of the Lebesgue integral, relative to *any* countably additive measure on *any* set. (The precise definitions follow.) This abstract theory is not in any way more difficult than the special case of the real line; it shows that a large part of integration theory is independent of any geometry (or topology) of the underlying space; and, of course, it gives us a tool of much wider applicability. The existence of a large class of measures, among them that of Lebesgue, will be established in Chap. 2.

Set-Theoretic Notations and Terminology

1.1 Some sets can be described by listing their members. Thus $\{x_1, \ldots, x_n\}$ is the set whose members are $x_1, \ldots, x_n$; and $\{x\}$ is the set whose only member is x. More often, sets are described by properties. We write

$$\{x: P\}$$

for the set of all elements x which have the property P. The symbol $\varnothing$ denotes the empty set. The words *collection*, *family*, and *class* will be used synonymously with *set*.

We write $x \in A$ if x is a member of the set A; otherwise $x \notin A$. If B is a subset of A, i.e., if $x \in B$ implies $x \in A$, we write $B \subset A$. If $B \subset A$ and $A \subset B$, then $A = B$. If $B \subset A$ and $A \neq B$, B is a *proper* subset of A. Note that $\varnothing \subset A$ for every set A.

$A \cup B$ and $A \cap B$ are the union and intersection of A and B, respectively. If $\{A_\alpha\}$ is a collection of sets, where α runs through some index set I, we write

$$\bigcup_{\alpha \in I} A_\alpha \qquad \text{and} \qquad \bigcap_{\alpha \in I} A_\alpha$$

for the union and intersection of $\{A_\alpha\}$:

$$\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for at least one } \alpha \in I\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for every } \alpha \in I\}.$$

If I is the set of all positive integers, the customary notations are

$$\bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} A_n.$$

If no two members of $\{A_\alpha\}$ have an element in common, then $\{A_\alpha\}$ is a *disjoint collection* of sets.

We write $A - B = \{x : x \in A, x \notin B\}$, and denote the complement of A by A^c whenever it is clear from the context with respect to which larger set the complement is taken.

The *cartesian product* $A_1 \times \cdots \times A_n$ of the sets $A_1, \ldots, A_n$ is the set of all ordered n-tuples $(a_1, \ldots, a_n)$ where $a_i \in A_i$ for $1 = 1, \ldots, n$.

The *real line* (or real number system) is R^1, and

$$R^k = R^1 \times \cdots \times R^1 \qquad (k \text{ factors}).$$

The *extended real number system* is R^1 with two symbols, ∞ and $-\infty$, adjoined, and with the obvious ordering. If $-\infty \le a \le b \le \infty$, the *interval* $[a, b]$ and the *segment* (a, b) are defined to be

$$[a, b] = \{x : a \le x \le b\}, \qquad (a, b) = \{x : a < x < b\}.$$

We also write

$$[a, b) = \{x : a \le x < b\}, \qquad (a, b] = \{x : a < x \le b\}.$$

If $E \subset [-\infty, \infty]$ and $E \neq \varnothing$, the least upper bound (supremum) and greatest lower bound (infimum) of E exist in $[-\infty, \infty]$ and are denoted by $\sup E$ and $\inf E$.

Sometimes (but only when $\sup E \in E$) we write $\max E$ for $\sup E$.

The symbol

$$f: X \to Y$$

means that f is a *function* (or *mapping* or *transformation*) of the set X into the set Y; i.e., f assigns to each $x \in X$ an element $f(x) \in Y$. If $A \subset X$ and $B \subset Y$, the *image* of A and the *inverse image* (or pre-image) of B are

$$f(A) = \{y : y = f(x) \text{ for some } x \in A\},$$

$$f^{-1}(B) = \{x : f(x) \in B\}.$$