Second edition

LINEAR ALGEBRA

KENNETH HOFFMAN / RAY KUNZE

LINEAR ALGEBRA

Second Edition

KENNETH HOFFMAN
Professor of Mathematics
Massachusetts Institute of Technology

RAY KUNZE
Professor of Mathematics
University of California, Irvine

PRENTICE-HALL, INC., Englewood Cliffs, New Jersey

PRENTICE-HALL INTERNATIONAL, INC., *London*
PRENTICE-HALL OF AUSTRALIA, PTY. LTD., *Sydney*
PRENTICE-HALL OF CANADA, LTD., *Toronto*
PRENTICE-HALL OF INDIA PRIVATE LIMITED, *New Delhi*
PRENTICE-HALL OF JAPAN, INC., *Tokyo*

Current printing (last digit):
10 9 8 7

Library of Congress Catalog Card No. 75-142120
Printed in the United States of America

Preface

Our original purpose in writing this book was to provide a text for the undergraduate linear algebra course at the Massachusetts Institute of Technology. This course was designed for mathematics majors at the junior level, although three-fourths of the students were drawn from other scientific and technological disciplines and ranged from freshmen through graduate students. This description of the M.I.T. audience for the text remains generally accurate today. The ten years since the first edition have seen the proliferation of linear algebra courses throughout the country and have afforded one of the authors the opportunity to teach the basic material to a variety of groups at Brandeis University, Washington University (St. Louis), and the University of California (Irvine).

Our principal aim in revising *Linear Algebra* has been to increase the variety of courses which can easily be taught from it. On one hand, we have structured the chapters, especially the more difficult ones, so that there are several natural stopping points along the way, allowing the instructor in a one-quarter or one-semester course to exercise a considerable amount of choice in the subject matter. On the other hand, we have increased the amount of material in the text, so that it can be used for a rather comprehensive one-year course in linear algebra and even as a reference book for mathematicians.

The major changes have been in our treatments of canonical forms and inner product spaces. In Chapter 6 we no longer begin with the general spatial theory which underlies the theory of canonical forms. We first handle characteristic values in relation to triangulation and diagonalization theorems and then build our way up to the general theory. We have split Chapter 8 so that the basic material on inner product spaces and unitary diagonalization is followed by a Chapter 9 which treats sesqui-linear forms and the more sophisticated properties of normal operators, including normal operators on real inner product spaces.

We have also made a number of small changes and improvements from the first edition. But the basic philosophy behind the text is unchanged.

We have made no particular concession to the fact that the majority of the students may not be primarily interested in mathematics. For we believe a mathematics course should not give science, engineering, or social science students a hodgepodge of techniques, but should provide them with an understanding of basic mathematical concepts.

On the other hand, we have been keenly aware of the wide range of backgrounds which the students may possess and, in particular, of the fact that the students have had very little experience with abstract mathematical reasoning. For this reason, we have avoided the introduction of too many abstract ideas at the very beginning of the book. In addition, we have included an Appendix which presents such basic ideas as set, function, and equivalence relation. We have found it most profitable not to dwell on these ideas independently, but to advise the students to read the Appendix when these ideas arise.

Throughout the book we have included a great variety of examples of the important concepts which occur. The study of such examples is of fundamental importance and tends to minimize the number of students who can repeat definition, theorem, proof in logical order without grasping the meaning of the abstract concepts. The book also contains a wide variety of graded exercises (about six hundred), ranging from routine applications to ones which will extend the very best students. These exercises are intended to be an important part of the text.

Chapter 1 deals with systems of linear equations and their solution by means of elementary row operations on matrices. It has been our practice to spend about six lectures on this material. It provides the student with some picture of the origins of linear algebra and with the computational technique necessary to understand examples of the more abstract ideas occurring in the later chapters. Chapter 2 deals with vector spaces, subspaces, bases, and dimension. Chapter 3 treats linear transformations, their algebra, their representation by matrices, as well as isomorphism, linear functionals, and dual spaces. Chapter 4 defines the algebra of polynomials over a field, the ideals in that algebra, and the prime factorization of a polynomial. It also deals with roots, Taylor's formula, and the Lagrange interpolation formula. Chapter 5 develops determinants of square matrices, the determinant being viewed as an alternating n-linear function of the rows of a matrix, and then proceeds to multilinear functions on modules as well as the Grassman ring. The material on modules places the concept of determinant in a wider and more comprehensive setting than is usually found in elementary textbooks. Chapters 6 and 7 contain a discussion of the concepts which are basic to the analysis of a single linear transformation on a finite-dimensional vector space; the analysis of characteristic (eigen) values, triangulable and diagonalizable transformations; the concepts of the diagonalizable and nilpotent parts of a more general transformation, and the rational and Jordan canonical forms. The primary and cyclic decomposition theorems play a central role, the latter being arrived at through the study of admissible subspaces. Chapter 7 includes a discussion of matrices over a polynomial domain, the computation of invariant factors and elementary divisors of a matrix, and the development of the Smith canonical form. The chapter ends with a discussion of semi-simple operators, to round out the analysis of a single operator. Chapter 8 treats finite-dimensional inner product spaces in some detail. It covers the basic geometry, relating orthogonalization to the idea of 'best approximation to a vector' and leading to the concepts of the orthogonal projection of a vector onto a subspace and the orthogonal complement of a subspace. The chapter treats unitary operators and culminates in the diagonalization of self-adjoint and normal operators. Chapter 9 introduces sesqui-linear forms, relates them to positive and self-adjoint operators on an inner product space, moves on to the spectral theory of normal operators and then to more sophisticated results concerning normal operators on real or complex inner product spaces. Chapter 10 discusses bilinear forms, emphasizing canonical forms for symmetric and skew-symmetric forms, as well as groups preserving non-degenerate forms, especially the orthogonal, unitary, pseudo-orthogonal and Lorentz groups.

We feel that any course which uses this text should cover Chapters 1, 2, and 3 thoroughly, possibly excluding Sections 3.6 and 3.7 which deal with the double dual and the transpose of a linear transformation. Chapters 4 and 5, on polynomials and determinants, may be treated with varying degrees of thoroughness. In fact, polynomial ideals and basic properties of determinants may be covered quite sketchily without serious damage to the flow of the logic in the text; however, our inclination is to deal with these chapters carefully (except the results on modules), because the material illustrates so well the basic ideas of linear algebra. An elementary course may now be concluded nicely with the first four sections of Chapter 6, together with (the new) Chapter 8. If the rational and Jordan forms are to be included, a more extensive coverage of Chapter 6 is necessary.

Our indebtedness remains to those who contributed to the first edition, especially to Professors Harry Furstenberg, Louis Howard, Daniel Kan, Edward Thorp, to Mrs. Judith Bowers, Mrs. Betty Ann (Sargent) Rose and Miss Phyllis Ruby. In addition, we would like to thank the many students and colleagues whose perceptive comments led to this revision, and the staff of Prentice-Hall for their patience in dealing with two authors caught in the throes of academic administration. Lastly, special thanks are due to Mrs. Sophia Koulouras for both her skill and her tireless efforts in typing the revised manuscript.

K. M. H. / R. A. K.

Contents

1. Linear Equations

1.1. Fields

We assume that the reader is familiar with the elementary algebra of real and complex numbers. For a large portion of this book the algebraic properties of numbers which we shall use are easily deduced from the following brief list of properties of addition and multiplication. We let F denote either the set of real numbers or the set of complex numbers.

1. Addition is commutative,

$$x + y = y + x$$

for all x and y in F.

2. Addition is associative,

$$x + (y + z) = (x + y) + z$$

for all x, y, and z in F.

3. There is a unique element 0 (zero) in F such that $x + 0 = x$, for every x in F.

4. To each x in F there corresponds a unique element $(-x)$ in F such that $x + (-x) = 0$.

5. Multiplication is commutative,

$$xy = yx$$

for all x and y in F.

6. Multiplication is associative,

$$x(yz) = (xy)z$$

for all x, y, and z in F.

7. There is a unique non-zero element 1 (one) in F such that $x1 = x$, for every x in F.

8. To each non-zero x in F there corresponds a unique element x^{-1} (or $1/x$) in F such that $xx^{-1} = 1$.

9. Multiplication distributes over addition; that is, $x(y + z) = xy + xz$, for all x, y, and z in F.

Suppose one has a set F of objects $x, y, z, \ldots$ and two operations on the elements of F as follows. The first operation, called addition, associates with each pair of elements x, y in F an element $(x + y)$ in F; the second operation, called multiplication, associates with each pair x, y an element xy in F; and these two operations satisfy conditions (1)–(9) above. The set F, together with these two operations, is then called a **field.** Roughly speaking, a field is a set together with some operations on the objects in that set which behave like ordinary addition, subtraction, multiplication, and division of numbers in the sense that they obey the nine rules of algebra listed above. With the usual operations of addition and multiplication, the set C of complex numbers is a field, as is the set R of real numbers.

For most of this book the 'numbers' we use may as well be the elements from any field F. To allow for this generality, we shall use the word 'scalar' rather than 'number.' Not much will be lost to the reader if he always assumes that the field of scalars is a subfield of the field of complex numbers. A **subfield** of the field C is a set F of complex numbers which is itself a field under the usual operations of addition and multiplication of complex numbers. This means that 0 and 1 are in the set F, and that if x and y are elements of F, so are $(x + y)$, $-x$, xy, and x^{-1} (if $x \neq 0$). An example of such a subfield is the field R of real numbers; for, if we identify the real numbers with the complex numbers $(a + ib)$ for which $b = 0$, the 0 and 1 of the complex field are real numbers, and if x and y are real, so are $(x + y)$, $-x$, xy, and x^{-1} (if $x \neq 0$). We shall give other examples below. The point of our discussing subfields is essentially this: If we are working with scalars from a certain subfield of C, then the performance of the operations of addition, subtraction, multiplication, or division on these scalars does not take us out of the given subfield.

EXAMPLE 1. The set of **positive integers**: 1, 2, 3, . . . , is not a subfield of C, for a variety of reasons. For example, 0 is not a positive integer; for no positive integer n is $-n$ a positive integer; for no positive integer n except 1 is $1/n$ a positive integer.

EXAMPLE 2. The set of **integers**: . . . , -2, -1, 0, 1, 2, . . . , is not a subfield of C, because for an integer n, $1/n$ is not an integer unless n is 1 or

-1. With the usual operations of addition and multiplication, the set of integers satisfies all of the conditions (1)–(9) except condition (8).

EXAMPLE 3. The set of **rational numbers,** that is, numbers of the form p/q, where p and q are integers and $q \neq 0$, is a subfield of the field of complex numbers. The division which is not possible within the set of integers is possible within the set of rational numbers. The interested reader should verify that any subfield of C must contain every rational number.

EXAMPLE 4. The set of all complex numbers of the form $x + y\sqrt{2}$, where x and y are rational, is a subfield of C. We leave it to the reader to verify this.

In the examples and exercises of this book, the reader should assume that the field involved is a subfield of the complex numbers, unless it is expressly stated that the field is more general. We do not want to dwell on this point; however, we should indicate why we adopt such a convention. If F is a field, it may be possible to add the unit 1 to itself a finite number of times and obtain 0 (see Exercise 5 following Section 1.2):

$$1 + 1 + \cdots + 1 = 0.$$

That does not happen in the complex number field (or in any subfield thereof). If it does happen in F, then the least n such that the sum of n 1's is 0 is called the **characteristic of the field** F. If it does not happen in F, then (for some strange reason) F is called a field of **characteristic zero.** Often, when we assume F is a subfield of C, what we want to guarantee is that F is a field of characteristic zero; but, in a first exposure to linear algebra, it is usually better not to worry too much about characteristics of fields.

1.2. Systems of Linear Equations

Suppose F is a field. We consider the problem of finding n scalars (elements of F) $x_1, \ldots, x_n$ which satisfy the conditions

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= y_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= y_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= y_m \end{aligned} \tag{1-1}$$

where $y_1, \ldots, y_m$ and A_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, are given elements of F. We call (1-1) a **system of m linear equations in n unknowns.** Any n-tuple $(x_1, \ldots, x_n)$ of elements of F which satisfies each of the equations in (1-1) is called a **solution** of the system. If $y_1 = y_2 = \cdots = y_m = 0$, we say that the system is **homogeneous,** or that each of the equations is homogeneous.

Perhaps the most fundamental technique for finding the solutions of a system of linear equations is the technique of elimination. We can illustrate this technique on the homogeneous system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 0 \\ x_1 + 3x_2 + 4x_3 &= 0. \end{aligned}$$

If we add (-2) times the second equation to the first equation, we obtain

$$-7x_2 - 7x_3 = 0$$

or, $x_2 = -x_3$. If we add 3 times the first equation to the second equation, we obtain

$$7x_1 + 7x_3 = 0$$

or, $x_1 = -x_3$. So we conclude that if (x_1, x_2, x_3) is a solution then $x_1 = x_2 = -x_3$. Conversely, one can readily verify that any such triple is a solution. Thus the set of solutions consists of all triples $(-a, -a, a)$.

We found the solutions to this system of equations by 'eliminating unknowns,' that is, by multiplying equations by scalars and then adding to produce equations in which some of the x_j were not present. We wish to formalize this process slightly so that we may understand why it works, and so that we may carry out the computations necessary to solve a system in an organized manner.

For the general system (1-1), suppose we select m scalars $c_1, \ldots, c_m$, multiply the jth equation by c_j and then add. We obtain the equation

$$(c_1A_{11} + \cdots + c_mA_{m1})x_1 + \cdots + (c_1A_{1n} + \cdots + c_mA_{mn})x_n = c_1y_1 + \cdots + c_my_m.$$

Such an equation we shall call a **linear combination** of the equations in (1-1). Evidently, any solution of the entire system of equations (1-1) will also be a solution of this new equation. This is the fundamental idea of the elimination process. If we have another system of linear equations

$$\begin{aligned} B_{11}x_1 + \cdots + B_{1n}x_n &= z_1 \\ \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ B_{k1}x_1 + \cdots + B_{kn}x_n &= z_k \end{aligned} \tag{1-2}$$

in which each of the k equations is a linear combination of the equations in (1-1), then every solution of (1-1) is a solution of this new system. Of course it may happen that some solutions of (1-2) are not solutions of (1-1). This clearly does not happen if each equation in the original system is a linear combination of the equations in the new system. Let us say that two systems of linear equations are **equivalent** if each equation in each system is a linear combination of the equations in the other system. We can then formally state our observations as follows.

Theorem 1. *Equivalent systems of linear equations have exactly the same solutions.*

If the elimination process is to be effective in finding the solutions of a system like (1-1), then one must see how, by forming linear combinations of the given equations, to produce an equivalent system of equations which is easier to solve. In the next section we shall discuss one method of doing this.

Exercises

1. Verify that the set of complex numbers described in Example 4 is a subfield of C.

2. Let F be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned} x_1 - x_2 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned} \qquad \begin{aligned} 3x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}$$

3. Test the following systems of equations as in Exercise 2.

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ \tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 \qquad - x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned}$$

4. Test the following systems as in Exercise 2.

$$\begin{aligned} 2x_1 + (-1 + i)x_2 \qquad + x_4 &= 0 \\ 3x_2 - 2ix_3 + 5x_4 &= 0 \end{aligned} \qquad \begin{aligned} \left(1 + \frac{i}{2}\right)x_1 + 8x_2 - ix_3 - x_4 &= 0 \\ \tfrac{2}{3}x_1 - \tfrac{1}{2}x_2 + x_3 + 7x_4 &= 0 \end{aligned}$$

5. Let F be a set which contains exactly two elements, 0 and 1. Define an addition and multiplication by the tables:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Verify that the set F, together with these two operations, is a field.

6. Prove that if two homogeneous systems of linear equations in two unknowns have the same solutions, then they are equivalent.

7. Prove that each subfield of the field of complex numbers contains every rational number.

8. Prove that each field of characteristic zero contains a copy of the rational number field.

1.3. Matrices and Elementary Row Operations

One cannot fail to notice that in forming linear combinations of linear equations there is no need to continue writing the 'unknowns' $x_1, \ldots, x_n$, since one actually computes only with the coefficients A_{ij} and the scalars y_i. We shall now abbreviate the system (1-1) by

$$AX = Y$$

where

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

We call A the **matrix of coefficients** of the system. Strictly speaking, the rectangular array displayed above is not a matrix, but is a representation of a matrix. An $m \times n$ **matrix over the field** F is a function A from the set of pairs of integers (i, j), $1 \leq i \leq m$, $1 \leq j \leq n$, into the field F. The **entries** of the matrix A are the scalars $A(i, j) = A_{ij}$, and quite often it is most convenient to describe the matrix by displaying its entries in a rectangular array having m rows and n columns, as above. Thus X (above) is, or defines, an $n \times 1$ matrix and Y is an $m \times 1$ matrix. For the time being, $AX = Y$ is nothing more than a shorthand notation for our system of linear equations. Later, when we have defined a multiplication for matrices, it will mean that Y is the product of A and X.

We wish now to consider operations on the rows of the matrix A which correspond to forming linear combinations of the equations in the system $AX = Y$. We restrict our attention to three **elementary row operations** on an $m \times n$ matrix A over the field F:

1. multiplication of one row of A by a non-zero scalar c;
2. replacement of the rth row of A by row r plus c times row s, c any scalar and $r \neq s$;
3. interchange of two rows of A.

An elementary row operation is thus a special type of function (rule) e which associated with each $m \times n$ matrix A an $m \times n$ matrix $e(A)$. One can precisely describe e in the three cases as follows:

1. $e(A)_{ij} = A_{ij}$ if $i \neq r$, $e(A)_{rj} = cA_{rj}$.
2. $e(A)_{ij} = A_{ij}$ if $i \neq r$, $e(A)_{rj} = A_{rj} + cA_{sj}$.
3. $e(A)_{ij} = A_{ij}$ if i is different from both r and s, $e(A)_{rj} = A_{sj}$, $e(A)_{sj} = A_{rj}$.

In defining $e(A)$, it is not really important how many columns A has, but the number of rows of A is crucial. For example, one must worry a little to decide what is meant by interchanging rows 5 and 6 of a 5×5 matrix. To avoid any such complications, we shall agree that an elementary row operation e is defined on the class of all $m \times n$ matrices over F, for some fixed m but any n. In other words, a particular e is defined on the class of all m-rowed matrices over F.

One reason that we restrict ourselves to these three simple types of row operations is that, having performed such an operation e on a matrix A, we can recapture A by performing a similar operation on $e(A)$.

Theorem 2. *To each elementary row operation* e *there corresponds an elementary row operation* e_1, *of the same type as* e, *such that* $e_1(e(A)) = e(e_1(A)) = A$ *for each* A. *In other words, the inverse operation (function) of an elementary row operation exists and is an elementary row operation of the same type.*

Proof. (1) Suppose e is the operation which multiplies the rth row of a matrix by the non-zero scalar c. Let e_1 be the operation which multiplies row r by c^{-1}. (2) Suppose e is the operation which replaces row r by row r plus c times row s, $r \neq s$. Let e_1 be the operation which replaces row r by row r plus $(-c)$ times row s. (3) If e interchanges rows r and s, let $e_1 = e$. In each of these three cases we clearly have $e_1(e(A)) = e(e_1(A)) = A$ for each A. ∎

Definition. *If* A *and* B *are* m × n *matrices over the field* F, *we say that* B **is row-equivalent to** A *if* B *can be obtained from* A *by a finite sequence of elementary row operations.*

Using Theorem 2, the reader should find it easy to verify the following. Each matrix is row-equivalent to itself; if B is row-equivalent to A, then A is row-equivalent to B; if B is row-equivalent to A and C is row-equivalent to B, then C is row-equivalent to A. In other words, row-equivalence is an equivalence relation (see Appendix).

Theorem 3. *If* A *and* B *are row-equivalent* m × n *matrices, the homogeneous systems of linear equations* AX = 0 *and* BX = 0 *have exactly the same solutions.*

Proof. Suppose we pass from A to B by a finite sequence of elementary row operations:

$$A = A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_k = B.$$

It is enough to prove that the systems $A_jX = 0$ and $A_{j+1}X = 0$ have the same solutions, i.e., that one elementary row operation does not disturb the set of solutions.

So suppose that B is obtained from A by a single elementary row operation. No matter which of the three types the operation is, (1), (2), or (3), each equation in the system $BX = 0$ will be a linear combination of the equations in the system $AX = 0$. Since the inverse of an elementary row operation is an elementary row operation, each equation in $AX = 0$ will also be a linear combination of the equations in $BX = 0$. Hence these two systems are equivalent, and by Theorem 1 they have the same solutions. ∎

EXAMPLE 5. Suppose F is the field of rational numbers, and

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}.$$

We shall perform a finite sequence of elementary row operations on A, indicating by numbers in parentheses the type of operation performed.

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 0 & -2 & -1 & 7 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 0 & 1 & \frac{1}{2} & -\frac{7}{2} \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 0 & -2 & 13 \\ 0 & 1 & \frac{1}{2} & -\frac{7}{2} \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 0 & 0 & \frac{15}{2} & -\frac{55}{2} \\ 1 & 0 & -2 & 13 \\ 0 & 1 & \frac{1}{2} & -\frac{7}{2} \end{bmatrix} \xrightarrow{(1)}$$

$$\begin{bmatrix} 0 & 0 & 1 & -\frac{11}{3} \\ 1 & 0 & -2 & 13 \\ 0 & 1 & \frac{1}{2} & -\frac{7}{2} \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 0 & 0 & 1 & -\frac{11}{3} \\ 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & \frac{1}{2} & -\frac{7}{2} \end{bmatrix} \xrightarrow{(2)}$$

$$\begin{bmatrix} 0 & 0 & 1 & -\frac{11}{3} \\ 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{5}{3} \end{bmatrix}$$

…lence of A with the final matrix in the above sequence …icular that the solutions of

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + 2x_4 &= 0 \\ x_1 + 4x_2 \qquad\quad - x_4 &= 0 \\ 2x_1 + 6x_2 - x_3 + 5x_4 &= 0 \end{aligned}$$

and

$$\begin{aligned} x_3 - \tfrac{11}{3}x_4 &= 0 \\ x_1 \qquad\qquad + \tfrac{17}{3}x_4 &= 0 \\ x_2 \qquad - \tfrac{5}{3}x_4 &= 0 \end{aligned}$$

are exactly the same. In the second system it is apparent that if we assign