PARTIAL DIFFERENTIAL EQUATIONS

G. F. D. DUFF

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BY

G. F. D. DUFF

ASSISTANT PROFESSOR OF MATHEMATICS
UNIVERSITY OF TORONTO

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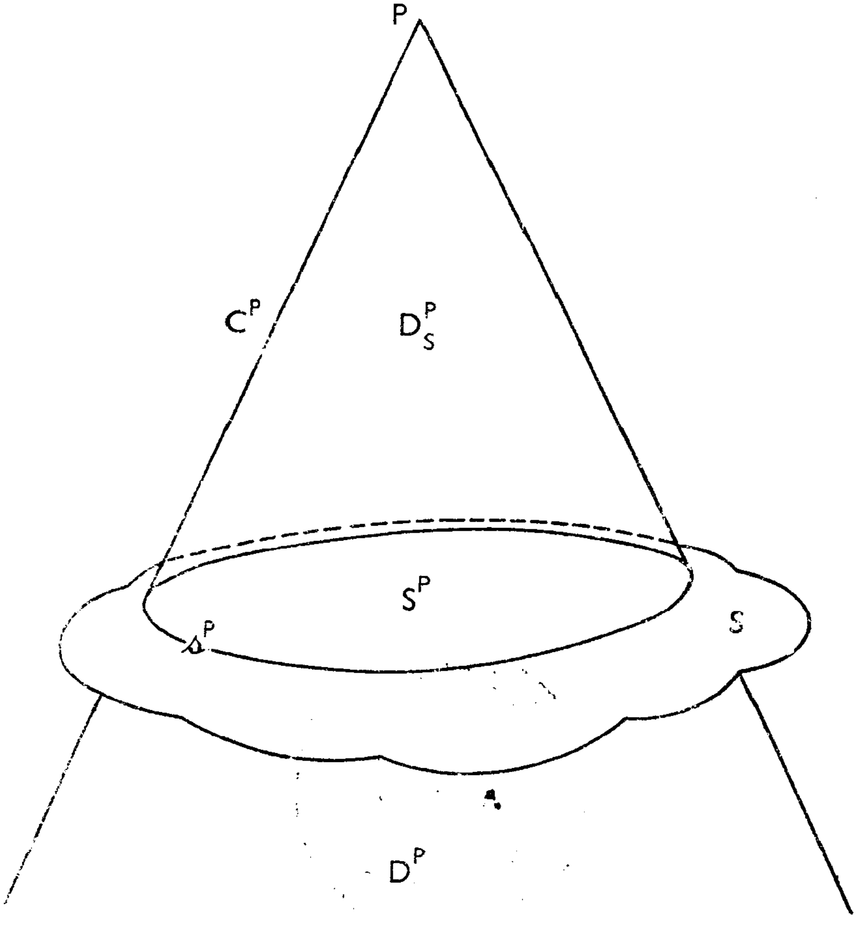

Cauchy's problem for the wave equation

PREFACE

THIS book is intended as an introduction to the theory of partial differential equations of the first and second orders, which will be useful to the prospective student of modern developments, and also to those who desire a detailed background for the traditional applications. The discussion is restricted to equations of the first order and linear equations of the second order, with one dependent variable, so that a large part of the work is severely classical. However, some of the material is relatively modern, though the methods and concepts used have been restricted to those of classical analysis. The notion of invariance under transformations of the independent variables is emphasized, and the exposition is self-contained as regards tensor calculus. The book might therefore be read by a student who has a good background in ordinary differential equations.

On the one hand, this treatment is intended to serve as preparation for such topics as general theories of the integration of differential systems, harmonic integrals, or the study of differential operators in function spaces. On the other hand, the equations treated here are useful in many branches of applied mathematics, and might well be studied from a more general standpoint by those who are concerned with the applications. As a branch of mathematical knowledge, our subject is notable for its many contacts with other branches of pure and applied mathematics. This interaction with its environment has produced not just a mass of diverse particular facts, but a well-organized and tightly knit theory. The aim is to present this aspect of the subject in a reasonably accessible form.

Each chapter begins with a summary wherein the motivation and order of topics within the chapter are described. Exercises are included in nearly every section; many of them are particular applications, though a few suggest further developments. A short bibliography, mainly of treatises and monographs, is appended, together with some references from each of the chapters. For the sake of brevity I have not attempted to state precise conditions of regularity for many of the results which are discussed herein, feeling that the student who wishes to pursue such questions will in any case turn to the standard treatises.

My colleagues Professors J. D. Burk and A. Robinson read the manuscript at various stages of completion and contributed many helpful criticisms, for which I make grateful acknowledgement. It is also a pleasure to thank Professor D. C. Spencer of Princeton University for several valuable comments.

G. F. D. DUFF

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I

DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

1.1. Some definitions and examples. A relation of the form

$$F\left(x^1, x^2,..., x^N, u, \frac{\partial u}{\partial x^1},..., \frac{\partial u}{\partial x^N}, \frac{\partial^2 u}{(\partial x^1)^2},..., \frac{\partial^k u}{(\partial x^N)^k}\right) = 0 \qquad (1.1.1)$$

is known as a partial differential equation. In this equation the quantities $x^1, x^2,..., x^N$ are independent variables; they range over certain sets of real values; or, as is often said for convenience, over a domain or region in the 'space' of the independent variables. A single set of numerical values $x^1,..., x^N$ is usually called a 'point' in that space. The quantity u which appears in the equation plays the role of a dependent variable. That is, u is assumed to be a function $u = f(x^1, x^2,..., x^N)$ of the independent variables, such that all derivatives of u which appear in (1.1.1) exist.

If we select a function $f(x^1, x^2,..., x^N)$ which possesses the requisite derivatives but is otherwise arbitrary, and if we form the expression

$$F\left(x^1, x^2,..., x^N, f(x^1,..., x^N), \frac{\partial f(x^1,..., x^N)}{\partial x^1},..., \frac{\partial^k f}{(\partial x^N)^k}\right), \qquad (1.1.2)$$

we obtain a compounded function which is a function of $x^1, x^2,..., x^N$. In general, this function will not be identically zero in the domain of the space of the $x^1,..., x^N$ which we wish to consider. However, if it does happen that the expression (1.1.2) vanishes identically in this domain, then we say that the function is a solution of the partial differential equation (1.1.1). That is, the relation

$$u = f(x^1,..., x^N), \qquad (1.1.3)$$

which is known as an integral relation or integral has as a consequence the truth of the differential relation (1.1.1) among the $x^1,..., x^N$, u, and the partial derivatives of u with respect to the $x^1,..., x^N$.

Throughout this book we shall limit the discussion to equations which, like (1.1.1), contain a single dependent variable u. The number N of independent variables will always be two or more, since if there were but one independent variable, the equation (1.1.1) would be an ordinary differential equation. We shall assume that the reader is familiar with certain properties of ordinary differential equations which, from time to time, will be needed in the exposition. The theory of partial differential equations is a natural extension of the theory of ordinary differential equations which in turn grew from the differential and integral calculus. The development of all these subjects has been powerfully stimulated by physical problems and applications which require analytical methods. Among these the questions which lead to partial differential equations are second to none in variety and usefulness. Our present purpose being, however, to study partial differential equations for their own sake, we shall employ an approach and motivation which is essentially mathematical.

We recall certain basic definitions. The order of a differential equation is the order of the highest derivative which appears in the equation. There are also definitions which refer to the algebraic structure of the equation. If the function F in (1.1.1) is linear in u and all of the derivatives of u which appear, the equation is said to be linear. Again, the linear equation is homogeneous if no term independent of u is present.

Linear homogeneous equations have a special property, which greatly facilitates their treatment, namely, that a constant multiple of a solution, or the sum of two or more solutions, is again a solution. Thus known solutions can be superposed, to build up new solutions. Not only sums, but integrals over solutions containing a parameter may be used, and for this reason it is often possible to find explicit formulae for the solutions of problems which involve linear equations. In comparison, the treatment of non-linear equations is much more difficult, and the available results less comprehensive.

Frequently it is possible to simplify the form of a differential equation by a suitable transformation of the dependent or independent variables. Certain standard forms have therefore been adopted and closely studied, since their properties carry over to equations of an apparently more general character. A well-known example of an equation in two independent variables (denoted by x and y for convenience) which may be written in either of two different forms is

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=f(x,y). \tag{1.1.4}$$

If in this equation we set $x+y=\xi$, $x-y=\eta$, then the rules for calculation with partial derivatives show that, as a function of ξ, η, the dependent variable u satisfies

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=\tfrac{1}{4}f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right) \tag{1.1.5}$$

When considering a differential equation such as (1.1.1), we must guard against the presumption that it has any solutions at all, until a proof of this has been supplied. In this connexion it might be necessary to specify the class of functions from which solutions will be selected. For example, the equation

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+1=0$$

clearly has no real-valued solutions whatever; but there are infinitely many complex-valued solutions of the form

$$u=\alpha x+\beta y+\gamma,$$

where α, β, γ are any complex constants such that $\alpha^2+\beta^2+1=0$. For simplicity, we shall hereafter assume that all equations, functions, and solutions are real-valued, unless the contrary is explicitly indicated.

Let us recall that an ordinary differential equation usually possesses infinitely many distinct solution functions. When the equation can be integrated in explicit analytical form, there appear constants of integration to which may be assigned numerical values in infinitely many ways. For instance, the equation of the second order $u''=f(x)$ has as solution

$$u(x)=\int\limits_0^x (x-t)f(t)\,dt+Ax+B,$$

where A and B are arbitrary constants. Thus the choice of a single solution function is made, in this case, by choosing two numbers. However, the solution of the most general form of a partial differential equation will usually contain arbitrary functions. Indeed, in the above example, if u depended on an additional variable y, the quantities A and B could be chosen as any functions of y.

Examples of the functional arbitrariness of general forms of solutions of partial differential equations are easily found. For example,

$$u=f(y-x^2),$$

where $f(t)$ is any once differentiable function, is a solution of the equation of the first order

$$\frac{\partial u}{\partial x}+2x\frac{\partial u}{\partial y}=0.$$

A general solution of the second-order equation (1.1.5) is

$$u(\xi,\eta)=\tfrac{1}{4}\int\limits_0^\xi\int\limits_0^\eta f\left(\frac{s+t}{2},\frac{s-t}{2}\right)dsdt+A(\xi)+B(\eta), \qquad (1.1.6)$$

in which there appear two arbitrary functions.

Conversely, it is often possible to derive a partial differential equation from a relation involving arbitrary functions. This can be done in the two cases above. As a further example consider the general equation

$$u = f(x^2+y^2)$$

of the surfaces of revolution about the u-axis in the Euclidean space of coordinates x, y, u. Suppose that f is differentiable; then we see easily that

$$y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y} = 0.$$

This partial differential equation of the first order characterizes the surfaces of revolution by a geometric property, namely, that the normal line to the surface meets the u-axis. Conversely, any integral of the partial differential equation has the form given above. The reader can easily verify this by transforming the equation to polar coordinates r, θ in the x, y plane.

Another equation of the first order which leads to a functional relation as integral is

$$\frac{\partial u}{\partial x}\frac{\partial g(x,y,u)}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial g(x,y,u)}{\partial x} = 0, \tag{1.1.7}$$

where $g(x,y,u)$ is a known function of the three variables x, y, and u. Let

$$f(x,y) = g\{x,y,u(x,y)\},$$

then the Jacobian

$$\begin{aligned}\frac{\partial(u,f)}{\partial(x,y)} &= \frac{\partial u}{\partial x}\frac{\partial f}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial f}{\partial x}\\ &= \frac{\partial u}{\partial x}\left(\frac{\partial g}{\partial y}+\frac{\partial g}{\partial u}\frac{\partial u}{\partial y}\right)-\frac{\partial u}{\partial y}\left(\frac{\partial g}{\partial x}+\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}\right)\\ &= \frac{\partial u}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial g}{\partial x}\end{aligned}$$

vanishes in view of the differential equation. Therefore there must be some functional relation independent of x and y which subsists between the two functions u and f of these two independent variables. Consequently any integral of the partial differential equation is defined implicitly by a relation

$$u = F\{g(x,y,u)\}.$$

The reader will easily verify that any such function u does satisfy the differential equation. Note that this equation is not linear.

Returning to equations of the second order, let us consider the homogeneous equation corresponding to (1.1.4), namely

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2} = 0. \tag{1.1.8}$$

This is the wave equation which governs the vibrations of a stretched string. From (1.1.6) and the transformation which leads from (1.1.4) to (1.1.5) we see that

$$u = f(x+y)+g(x-y) \tag{1.1.9}$$

is the most general solution of the equation. If, say, y is interpreted as a time variable, as is appropriate for the string, the two terms of this solution represent waves of arbitrary form but fixed velocity, one wave travelling in each direction along the string.

If we admit complex numbers and replace y by iy ($i^2 = -1$) in (1.1.8) we see that the equation becomes

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0, \tag{1.1.10}$$

and has the solution

$$u = f(x+iy)+g(x-iy). \tag{1.1.11}$$

This form of solution suggests the introduction of a complex variable $z = x+iy$, and of its complex conjugate $\bar{z} = x-iy$. It may be remembered that the real and imaginary parts of an analytic function

$$f(z) = f(x+iy) = u+iv$$

separately satisfy (1.1.10). This is Laplace's equation in the plane, and, partly because of the connexion with complex variable theory just suggested, it has been studied in greater detail than any other partial differential equation.

Exercise 1. Eliminate the arbitrary functions from

(*a*) $u = x^k f(y/x)$,

(*b*) $u = f(x^2+y^2+u^2)$,

(*c*) $u = f(x+\alpha y)+g(x+\beta y)$,

(*d*) $u = f(x+\alpha y)+xg(x+\alpha y)$.

(*e*) $u = f(x\cos\alpha + y\sin\alpha + z) + g(x\cos\alpha + y\sin\alpha - z)$.

Exercise 2. Find a solution, containing three disposable functions, of

$$\frac{\partial^3 u}{\partial x(\partial y)^2}-\frac{\partial^3 u}{\partial x^3} = 0.$$

Exercise 3. Find a general form of solution for the equation in three variables

$$\frac{\partial^3 u}{\partial x\partial y\partial z} = f(x,y,z).$$

Exercise 4. The envelope of a one-parameter family of planes in Euclidean space of the variables x, y, u, satisfies

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 = 0.$$

Hint: Write the family of planes

$$u = \lambda x+f(\lambda)y+g(\lambda),$$

differentiate with respect to λ, and show that

$$\frac{\partial u}{\partial x} = f\left(\frac{\partial u}{\partial y}\right).$$

Exercise 5. Separation of variables. The heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

has solutions of the form $u = \varphi(t)\psi(x)$ where

$$\varphi(t) = \exp\{-\lambda^2(t-\tau)\},$$
$$\psi(x) = \sin\lambda(x-\xi),\ \cos\lambda(x-\xi),$$

where λ, τ, ξ are any constants. Hence, by superposition, the integral

$$u = \int\limits_{-\infty}^{\infty} f(\lambda)e^{-\lambda^2 t}\cos(\lambda x)\,d\lambda$$

is a solution. When $f(\lambda) = 1$, $u = \left(\frac{\pi}{t}\right)^{\frac{1}{2}}\exp\left(-\frac{x^2}{4t}\right)$, $t > 0$.

Exercise 6. Separation of variables. The equation of the first order

$$f_1(x)\left(\frac{\partial u}{\partial x}\right)^2+f_2(y)\left(\frac{\partial u}{\partial y}\right)^2 = g_1(x)+g_2(y)$$

has solutions of the form $u = \varphi(x)+\psi(y)$. Find a family of these solutions which depends upon two arbitrary constants.

Exercise 7. Show that the equation $u_t+uu_x = u_{xx}$ can be transformed to the linear form $z_t = z_{xx}$ by introducing a new dependent variable z defined by the relation $2u = -(\log z)_x$.

1.2. The classification of equations and their solutions. The earliest and most obvious classification of partial differential equations was made on a formal basis. Such properties as the number of independent or dependent variables, the order of the equation, and its algebraic or functional structure, are necessarily the first to be considered. Further classifications have been made according to the methods which are available to treat special types of partial differential equations. For instance, linear equations with constant coefficients may be said to form a category of equations of this sort, since there are methods for finding explicit solutions of these equations, which methods fail when applied to equations with variable coefficients.

However, there is still another important principle by which partial differential equations may be classified. In all studies of this subject, it is really the properties of solutions of the equations which are important, rather than formal properties of the equation itself. In many cases the solutions of equations formally quite similar have radically different properties, and the equations should therefore be treated as distinct types. Classifications based on properties of the solutions are usually less obvious and also deeper than those based on structural properties of the equations. Probably the most important example of such a classification is the division into types of linear equations of the second order considered in Chapter IV.

It is desirable that a classification should take some account of the manner in which equations can be transformed by a change of independent or dependent variables. If the properties of solutions determine the classification, such a transformation will not alter the type of the partial differential equation.

In most practical applications, as well as in many theoretical problems, it is required to find one particular solution which also satisfies certain definite additional conditions. These auxiliary conditions may take the form of 'initial' or 'boundary' conditions, or both. It happens that certain types of auxiliary conditions are appropriate only to certain corresponding types of partial differential equations, in the sense that only for these equations can a solution satisfying the boundary conditions be proved to exist.

The problems of mathematical physics which lead to partial differential equations usually suggest appropriate auxiliary conditions at the same time. For instance, the problems of potential theory (Newtonian gravitation) lead to elliptic equations and to Dirichlet's problem in which boundary conditions are specified on a closed curve or surface; and this type of problem is appropriate for elliptic equations. Similarly, many problems of wave motion lead to hyperbolic equations and to auxiliary initial conditions which are meaningful for hyperbolic equations. When auxiliary conditions are specified in such a way that there exists one and only one solution of the partial differential equation which satisfies the conditions, then the problem of finding the solution is said to be *correctly set*. It is natural that practical problems should be a guide in suggesting how auxiliary conditions may be found for a given partial differential equation. In many cases when this guide is not available, correct auxiliary conditions are not known. However, such considerations as these, though they often suggest the result to be established, do not provide rigorous proofs that the solution of the problem exists.

In connexion with auxiliary conditions it is necessary to consider not only how many functions are to be specified, and how they are to determine the solution, but also such properties of these functions as continuity or analyticity. The data of physical problems are never exact; we must determine the effect upon the solution of small variations or uncertainties in the specified functions which appear in the auxiliary conditions. If small but arbitrary changes in the data lead to equally small perturbations of the solution, the problem is stable, otherwise it is unstable.

The analytical nature of the solution desired is an important part of any problem in partial differential equations. Both the intrinsic nature of the solution, its properties of continuity and differentiability, and the form in which it is to be expressed may determine what method is most appropriate for the problem at hand. Often the local behaviour of the solution can be deduced from the form of the differential equation, or from known existence theorems. Such knowledge can also influence the form in which the solution functions may be expressed—whether by known functions, series, integrals, implicit functions, or other means. If, for example, a solution is known to possess discontinuities, it is not possible to represent these by using a power series expansion.

Among the functions commonly used to represent mathematical or physical phenomena, those which are analytic stand out as a special class, because of their 'rigid' nature. By an analytic function we mean a real-valued function of real variables, expressible by means of convergent power series expansions in those variables. The 'rigidity' of analytic functions springs from the fact that if such a function, together with all its derivatives, is known at one point, then its values at all other points are fixed by the process of continuation with power series. If a change in value, however small, is made at one point, the values of the analytic function at all other points will, in general, be affected. Thus analytic functions are not very suitable for the representation of physical phenomena, in which events at different points may be quite independent of each other. Most of the partial differential equations which appear in practice are themselves analytic, that is, all functions and coefficients in them are analytic in their various arguments; but by no means are all solutions of all these equations also analytic.

Another feature of the study of solutions of partial differential equations deserves our attention, namely, the division of the theory into its local and global aspects. A differential equation is a statement regarding the value of a function and its derivatives at a point—that is, of the function in an arbitrarily small neighbourhood of the point—and many properties of solutions which can be deduced directly from the equation are local properties such as smoothness and regularity. In the next section will be presented existence theorems, applicable to very general analytic equations, which show that these equations do possess solutions defined in small but finite regions—the neighbourhood of a point or of a surface, for example. These results must also be regarded as local.

To solve most practical questions, however, it is necessary to find solutions defined in a given region of finite, or perhaps even infinite, extent. This is the global or 'in the large' aspect. One approach to it would be to piece together solutions which are defined locally; but this is often not feasible, so that quite different methods are needed. In a certain sense, a problem 'in the large' requires us to have at hand all solutions of the partial differential equation, and to select, by some means, the appropriate one. For this reason we may wish to characterize, or if possible to construct, the whole 'manifold of solutions' of a given equation. With ordinary differential equations, explicit integration is usually the difficult process, while the fitting of boundary conditions is easier. The reverse is often true for partial differential equations—it is not hard to find a general form of solution, but it is difficult to specialize it in order to satisfy auxiliary conditions.

These various concepts may be illustrated by a comparison of the two second-order equations mentioned in § 1, namely the wave equation, and Laplace's equation in two dimensions. The wave equation in the independent variables x, t is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \tag{1.2.1}$$

and it has the general solution

$$u = f(x+t)+g(x-t). \tag{1.2.2}$$

The functions f and g should be assumed twice differentiable, in order that the second derivatives which appear in the equation (1.2.1) should exist.