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Equilibrium States
Models in Quantum Statistical Mechanics

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States in Quantum Statistical Mechanics

5.1. Introduction

In this chapter, and the following one, we examine various applications of C^*-algebras and their states to statistical mechanics. Principally we analyze the structural properties of the equilibrium states of quantum systems consisting of a large number of particles. In Chapter 1 we argued that this leads to the study of states of infinite-particle systems as an initial approximation. There are two approaches to this study which are to a large extent complementary.

The first approach begins with the specific description of finite systems and their equilibrium states provided by quantum statistical mechanics. One then rephrases this description in an algebraic language which identifies the equilibrium states as states over a quasi-local C^*-algebra generated by subalgebras corresponding to the observables of spatial subsystems. Finally, one attempts to calculate an approximation of these states by taking their limit as the volume of the system tends to infinity, the so-called thermodynamic limit. The infinite-volume equilibrium states obtained in this manner provide the data for the calculation of bulk properties of the matter under consideration as functions of the thermodynamic variables. By this we mean properties such as the particle density, or specific heat, as functions of the temperature and chemical potential, etc. In fact, the infinite-volume data provides a much more detailed, even microscopic, description of the equilibrium phenomena although one is only generally interested in the bulk properties and their fluctuations. Examination of the thermodynamic limit also provides a test of the scope of the usual statistical mechanical formalism. If this formalism is rich enough to describe phase transitions, then at certain critical values of the thermodynamic parameters there should be a multiplicity of infinite-volume limit states arising from slight variations of the external interactions or boundary conditions. These states would correspond to various phases and mixtures of these phases. In such a situation it should be possible to arrange the limits such that phase separation takes place and then the equilibrium states would also provide information concerning interface phenomena such as surface tension.

The second approach to algebraic statistical mechanics avoids discussion of the thermodynamic limit and attempts to characterize and classify the equilibrium states of the infinite system as states over an appropriate C^*-algebra. The elements of the C^*-algebra represent kinematic observables, i.e., observables at a given time, and the states describe the instantaneous

states of the system. For a complete physical description it is necessary to specify the dynamical law governing the change with time of the observables, or the states, and the equilibrium states are determined by their properties with respect to this dynamics. The general nature of the dynamical law can be inferred from the usual quantum-mechanical formalism and it appears that there are various possibilities. Recall that for finite quantum systems the dynamics is given by a one-parameter group of *-automorphisms of the algebra of observables,

$$A \mapsto \tau_t(A) = e^{itH} A e^{-itH},$$

where H is the selfadjoint Hamiltonian operator of the system. Thus it appears natural that the dynamics of the infinite system should be determined by a continuous one-parameter group of *-automorphisms τ of the C^*-algebra of observables. This type of dynamics is certainly the simplest possible and it occurs in various specific models, e.g., the noninteracting Fermi gas, some of which we examine in the sequel. Nevertheless, it is not the general situation. The difficulty is that a group of this kind automatically defines a continuous development of every state of the system. But this is not to be expected for general infinite systems in which complicated phenomena involving the local accumulation of an infinite number of particles and energy can occur for certain initial states. Thus it is necessary to examine weaker forms of evolution. For example, one could assume the dynamics to be specified as a group of automorphisms of the von Neumann algebras corresponding to a subclass of states over the C^*-algebra. Alternatively one could adopt an infinitesimal description and assume that the evolution is determined by a derivation which generates an automorphism group only in certain representations. Each of these possible structures could in principle be verified in a particular model by a thermodynamic limiting process and each such structure provides a framework for characterizing equilibrium phenomena. To understand the type of characterization which is possible it is useful to refer to the finite-volume description of equilibrium.

There are various possible descriptions of equilibrium states, which all stem from the early work of Boltzmann and Gibbs on classical statistical mechanics, and which differ only in their initial specification. The three most common possibilities are the microcanonical ensemble, the canonical ensemble, and the grand canonical ensemble. In the first, the energy and particle number are held fixed; in the second, states of various energy are allowed for fixed particle number; and in the third, both the energy and the particle number vary. Each of these descriptions can be rephrased algebraically but the grand canonical description is in several ways more convenient. Let $\mathfrak{H}$ be the Hilbert space of states for all possible energies and particle numbers of the finite system, and H and N, the selfadjoint Hamiltonian and number operators, respectively. The Gibbs grand canonical equilibrium state is defined as a state over $\mathscr{L}(\mathfrak{H})$, or $\mathscr{L}\mathscr{C}(\mathfrak{H})$, by

$$\omega_{\beta,\mu}(A) = \frac{\mathrm{Tr}_{\mathfrak{H}}(e^{-\beta K} A)}{\mathrm{Tr}_{\mathfrak{H}}(e^{-\beta K})},$$

where $K = H - \mu N$, β, $\mu \in \mathbb{R}$, and it is assumed that $e^{-\beta K}$ is a trace-class operator. Typically H is lower semi-bounded and the trace-class property is valid for all $\beta > 0$. The parameters β and μ correspond to the inverse temperature of the system, in suitable units, and the chemical potential, respectively, and therefore this description is well-suited to a given type of material at a fixed temperature. Now if the generalized evolution τ is defined by

$$A \in \mathscr{L}(\mathfrak{H}) \mapsto \tau_t(A) = e^{itK} A e^{-itK} \in \mathscr{L}(\mathfrak{H}),$$

then the trace-class property of $e^{-\beta K}$ allows one to deduce that the functions

$$t \mapsto \omega_{\beta,\mu}(A\tau_t(B))$$

are analytic in the open strip $0 < \operatorname{Im} t < \beta$ and continuous on the boundaries of the strip. Moreover, the cyclicity of the trace gives

$$\omega_{\beta,\mu}(A\tau_t(B))|_{t=i\beta} = \omega_{\beta,\mu}(BA).$$

This is the KMS condition which we briefly described in Chapter 1 and which will play an important role throughout this chapter. One significance of this condition is that it uniquely determines the Gibbs state over $\mathscr{L}\mathscr{C}(\mathfrak{H})$, i.e., the only state over $\mathscr{L}\mathscr{C}(\mathfrak{H})$ which satisfies the KMS condition with respect to τ at the value β is the Gibbs grand canonical equilibrium state. This can be proved by explicit calculation but it will in fact follow from the characterization of extremal KMS states occurring in Section 5.3. It also follows under quite general conditions that the KMS condition is stable under limits. Thus for a system whose kinematic observables form a C^*-algebra $\mathfrak{A}$ and whose dynamics is supposed to be given by a continuous group of *-automorphisms τ of $\mathfrak{A}$, it is natural to take the KMS condition as an empirical definition of an equilibrium state.

Prior to the analysis of KMS states we introduce the specific quasi-local C^*-algebras which provide the quantum-mechanical description of systems of point particles and examine various properties of their states and representations. In particular we discuss the equilibrium states of systems of non-interacting particles. This analysis illustrates the thermodynamic limiting process, utilizes the KMS condition as a calculational device, and also provides a testing ground for the general formalism which we subsequently develop.

In the latter half of the chapter we discuss attempts to derive the KMS condition from first principles.

5.2. Continuous Quantum Systems. I

5.2.1. The CAR and CCR Relations

There are two approaches to the algebraic structure associated with systems of point particles in quantum mechanics. The first is quite concrete and physical. One begins with the Hilbert space of vector states of the particles and subsequently introduces algebras of operators corresponding to certain particle observables. The second approach is more abstract and consists of postulating certain structural features of a C^*-algebra of observables and then proving uniqueness of the algebra. One recovers the first point of view by passing to a particular representation. We discuss the first concrete approach in this subsection and then in Section 5.2.2 we examine the abstract formulation.

The quantum-mechanical states of n identical point particles in the configuration space $\mathbb{R}^\nu$ are given by vectors of the Hilbert space $L^2(\mathbb{R}^{n\nu})$. If the number of particles is not fixed, the states are described by vectors of the direct sum space

$$\mathfrak{F} = \bigoplus_{n \geq 0} L^2(\mathbb{R}^{n\nu}),$$

i.e., sequences $\psi = \{\psi^{(n)}\}_{n \geq 0}$, where $\psi^{(0)} \in \mathbb{C}$, $\psi^{(n)} \in L^2(\mathbb{R}^{n\nu})$ for $n \geq 1$, and the norm of ψ is given by

$$\|\psi\|^2 = |\psi^{(0)}|^2 + \sum_{n \geq 1} \int dx_1 \cdots dx_n |\psi^{(n)}(x_1, \ldots, x_n)|^2.$$

There is, however, a further restriction imposed by quantum statistics.

If $\psi \in \mathfrak{F}$ is normalized, then

$$d\rho(x_1, \ldots, x_n) = |\psi^{(n)}(x_1, \ldots, x_n)|^2 dx_1 \cdots dx_n$$

is the quantum-mechanical probability density for ψ to describe n particles at the infinitesimal neighborhood of the points $x_1, \ldots, x_n$. The normalization of ψ corresponds to the normalization of the total probability to unity. But in microscopic physics identical particles are indistinguishable and this is reflected by the symmetry of the probability density under interchange of the particle coordinates. This interchange defines a unitary representation of the permutation group and the symmetry is assured if the ψ transform under a suitable subrepresentation. There are two cases of paramount importance.

The first arises when the components $\psi^{(n)}$ of each ψ are symmetric under interchange of coordinates. Particles whose states transform in this manner are called *bosons* and are said to satisfy *Bose (–Einstein) statistics*. The second case corresponds to anti-symmetry of the $\psi^{(n)}$ under interchange of each pair of coordinates. The associated particles are called *fermions* and are said to satisfy *Fermi (–Dirac) statistics*. Thus to discuss these two types of particle one must examine the Hilbert subspaces $\mathfrak{F}_\pm$, of $\mathfrak{F}$, formed by the $\psi = \{\psi^{(n)}\}_{n\geq 0}$ whose components are symmetric (the + sign) or anti-symmetric (the − sign). These subspaces are usually called Fock spaces but we will also use the term for more general direct sum spaces.

To describe particles which have internal structure, e.g., an intrinsic angular momentum, or spin, it is necessary to generalize the above construction of Fock space.

Assume that the states of each particle form a complex Hilbert space $\mathfrak{h}$ and let $\mathfrak{h}^n = \mathfrak{h} \otimes \mathfrak{h} \otimes \cdots \otimes \mathfrak{h}$ denote the n-fold tensor product of $\mathfrak{h}$ with itself. Further introduce the *Fock space* $\mathfrak{F}(\mathfrak{h})$ by

$$\mathfrak{F}(\mathfrak{h}) = \bigoplus_{n\geq 0} \mathfrak{h}^n,$$

where $\mathfrak{h}^0 = \mathbb{C}$. Thus a vector $\psi \in \mathfrak{F}(\mathfrak{h})$ is a sequence $\{\psi^{(n)}\}_{n\geq 0}$ of vectors $\psi^{(n)} \in \mathfrak{h}^n$ and $\mathfrak{h}^n$ can be identified as the closed subspace of $\mathfrak{F}(\mathfrak{h})$ formed by the vectors with all components except the nth equal to zero.

In order to introduce the subspaces relevant to the description of bosons and fermions we first define operators $P_\pm$ on $\mathfrak{F}(\mathfrak{h})$ by

$$P_+(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = (n!)^{-1} \sum_\pi f_{\pi_1} \otimes f_{\pi_2} \otimes \cdots \otimes f_{\pi_n},$$

$$P_-(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = (n!)^{-1} \sum_\pi \varepsilon_\pi f_{\pi_1} \otimes f_{\pi_2} \otimes \cdots \otimes f_{\pi_n}$$

for all $f_1, \ldots, f_n \in \mathfrak{h}$. The sum is over all permutations π; $(1, 2, \ldots, n) \mapsto (\pi_1, \pi_2, \ldots, \pi_n)$ of the indices and ε_π is one if π is even and minus one if π is odd. Extension by linearity yields two densely defined operators with $\|P_\pm\| = 1$ and the $P_\pm$ extend by continuity to bounded operators of norm one. The P_+ and P_-, restricted to $\mathfrak{h}^n$, are the projections onto the subspaces of $\mathfrak{h}^n$ corresponding to the one-dimensional unitary representations $\pi \mapsto 1$ and $\pi \mapsto \varepsilon_\pi$ of the permutation group of n elements, respectively. The *Bose–Fock space* $\mathfrak{F}_+(\mathfrak{h})$ and the *Fermi–Fock space* $\mathfrak{F}_-(\mathfrak{h})$ are then defined by

$$\mathfrak{F}_\pm(\mathfrak{h}) = P_\pm \mathfrak{F}(\mathfrak{h})$$

and the corresponding n-particle subspaces $\mathfrak{h}^n_\pm$ by $\mathfrak{h}^n_\pm = P_\pm \mathfrak{h}^n$. We also define a *number operator* N on $\mathfrak{F}(\mathfrak{h})$ by

$$D(N) = \left\{\psi; \psi = \{\psi^{(n)}\}_{n\geq 0}, \sum_{n\geq 0} n^2 \|\psi^{(n)}\|^2 < +\infty\right\}$$

and

$$N\psi = \{n\psi^{(n)}\}_{n\geq 0}$$

for each $\psi \in D(N)$. It is evident that N is selfadjoint since it is already given in its spectral representation. Note that e^{itN} leaves the subspaces $\mathfrak{F}_{\pm}(\mathfrak{h})$ invariant. We will also use N to denote the selfadjoint restrictions of the number operator to these subspaces.

The peculiar structure of Fock space allows the amplification of operators on $\mathfrak{h}$ to the whole spaces $\mathfrak{F}_{\pm}(\mathfrak{h})$ by a method commonly referred to as *second quantization*. This is of particular interest for selfadjoint operators and unitaries.

If H is a selfadjoint operator on $\mathfrak{h}$, one can define H_n on $\mathfrak{h}_{\pm}^n$ by setting $H_0 = 0$ and

$$H_n(P_{\pm}(f_1 \otimes \cdots \otimes f_n)) = P_{\pm}\left(\sum_{i=1}^{n} f_1 \otimes f_2 \otimes \cdots \otimes Hf_i \otimes \cdots \otimes f_n\right)$$

for all $f_i \in D(H)$, and then extending by continuity. The direct sum of the H_n is essentially selfadjoint because (1) it is symmetric and hence closable, (2) it has a dense set of analytic vectors formed by finite sums of (anti-) symmetrized products of analytic vectors of H. The selfadjoint closure of this sum is called the second quantization of H and is denoted by $d\Gamma(H)$. Thus

$$d\Gamma(H) = \overline{\bigoplus_{n \geq 0} H_n}.$$

The simplest example of this second quantization is given by choosing $H = \mathbb{1}$, one then has

$$d\Gamma(\mathbb{1}) = N.$$

If U is unitary, U_n is defined by $U_0 = \mathbb{1}$ and by setting

$$U_n(P_{\pm}(f_1 \otimes f_2 \otimes \cdots \otimes f_n)) = P_{\pm}(Uf_1 \otimes Uf_2 \otimes \cdots \otimes Uf_n)$$

and extending by continuity. The second quantization of U is denoted by $\Gamma(U)$, where

$$\Gamma(U) = \bigoplus_{n \geq 0} U_n.$$

Note that $\Gamma(U)$ is unitary. The notation $d\Gamma$ and Γ is chosen because if $U_t = e^{itH}$ is a strongly continuous one-parameter unitary group, then

$$\Gamma(U_t) = e^{it d\Gamma(H)}.$$

Next we wish to describe two C^*-algebras of observables associated with bosons and fermions, respectively. Both algebras are defined with the aid of particle "*annihilation*" and "*creation*" *operators* which are introduced as follows. For each $f \in \mathfrak{h}$ we define operators $a(f)$, and $a^*(f)$, on $\mathfrak{F}(\mathfrak{h})$ by initially setting $a(f)\psi^{(0)} = 0$, $a^*(f)\psi^{(0)} = f$, $f \in \mathfrak{h}$, and

$$a(f)(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = n^{1/2}(f, f_1) f_2 \otimes f_3 \otimes \cdots \otimes f_n,$$
$$a^*(f)(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = (n+1)^{1/2} f \otimes f_1 \otimes \cdots \otimes f_n.$$

Extension by linearity again yields two densely defined operators and if $\psi^{(n)} \in \mathfrak{h}^n$, one easily calculates that

$$\|a(f)\psi^{(n)}\| \leq n^{1/2}\|f\|\|\psi^{(n)}\|, \qquad \|a^*(f)\psi^{(n)}\| \leq (n+1)^{1/2}\|f\|\|\psi^{(n)}\|.$$

Thus $a(f)$ and $a^*(f)$ have well-defined extensions to the domain $D(N^{1/2})$ of $N^{1/2}$ and

$$\|a^\#(f)\psi\| \leq \|f\|\|(N+1)^{1/2}\psi\|$$

for all $\psi \in D(N^{1/2})$, where $a^\#(f)$ denotes either $a(f)$ or $a^*(f)$. Moreover, one has the adjoint relation

$$(a^*(f)\varphi, \psi) = (\varphi, a(f)\psi)$$

for all $\varphi, \psi \in D(N^{1/2})$. Finally, we define annihilation and creation operators $a_\pm(f)$ and $a_\pm^*(f)$ on the Fock spaces $\mathfrak{F}_\pm(\mathfrak{h})$ by

$$a_\pm(f) = P_\pm a(f)P_\pm, \qquad a_\pm^*(f) = P_\pm a^*(f)P_\pm.$$

The relations

$$(a_\pm^*(f)\varphi, \psi) = (\varphi, a_\pm(f)\psi), \qquad \|a_\pm^*(f)\psi\| \leq \|f\|\|(N+1)^{1/2}\psi\|$$

follow from the corresponding relations for $a(f)$ and $a^*(f)$. Moreover,

$$a_\pm(f) = a(f)P_\pm, \qquad a_\pm^*(f) = P_\pm a^*(f)$$

because $a(f)$ leaves the subspaces $\mathfrak{F}_\pm(\mathfrak{h})$ invariant. Note that the maps $f \mapsto a_\pm(f)$ are anti-linear but the maps $f \mapsto a_\pm^*(f)$ are linear.

The physical interpretation of these operators is the following. Let $\Omega = (1, 0, 0, \ldots)$, then Ω corresponds to the zero-particle state, the *vacuum*. The vectors

$$\psi_\pm(f) = a_\pm^*(f)\Omega$$

identify with elements of the one-particle space $\mathfrak{h}$ and hence $a_\pm^*(f)$ "creates" a particle in the state f. The vectors

$$\begin{aligned}\psi_\pm(f_1, \ldots, f_n) &= (n!)^{-1/2}a_\pm^*(f_1)\cdots a_\pm^*(f_n)\Omega \\ &= P_\pm(f_1 \otimes \cdots \otimes f_n)\end{aligned}$$

are n-particle states which arise from successive "creation" of particles in the states $f_n, f_{n-1}, \ldots, f_1$. Similarly the $a_\pm(f)$ reduce the number of particles, i.e., they annihilate particles. Note that if $f_i = f_j$ for some pair i, j with $1 \leq i < j \leq n$, then

$$\psi_-(f_1, \ldots, f_n) = P_-(f_1 \otimes \cdots \otimes f_n) = 0$$

by anti-symmetry. Thus it is impossible to create two fermions in the same state. This is the celebrated *Pauli principle* which is reflected by the operator equation

$$a_-^*(f)a_-^*(f) = 0.$$

This last relation is the simplest case of the commutation relations which link the annihilation and creation operators.

One computes straightforwardly that

$$[a_+(f), a_+(g)] = 0 = [a_+^*(f), a_+^*(g)],$$
$$[a_+(f), a_+^*(g)] = (f, g)\mathbb{1},$$

and

$$\{a_-(f), a_-(g)\} = 0 = \{a_-^*(f), a_-^*(g)\},$$
$$\{a_-(f), a_-^*(g)\} = (f, g)\mathbb{1},$$

where we have again used the notation $\{A, B\} = AB + BA$. The first relations are called the *canonical commutation relations* (CCRs) and the second the *canonical anti-commutation relations* (CARs).

Although there is a superficial similarity between these two sets of algebraic rules, the properties of the respective operators are radically different. In applications to physics these differences are thought to be at the root of the fundamentally disparate behaviors of Bose and Fermi systems at low temperatures. In order to emphasize these differences we separate the subsequent discussion of the CARs and CCRs but before the general analysis we give an example of the creation and annihilation operators for point particles.

EXAMPLE 5.2.1. If $\mathfrak{h} = L^2(\mathbb{R}^\nu)$, then $\mathfrak{F}_\pm(\mathfrak{h})$ consists of sequences $\{\psi^{(n)}\}_{n\geq 0}$ of functions of n variables $x_i \in \mathbb{R}^\nu$ which are totally symmetric (+ sign) or totally anti-symmetric (− sign). The action of the annihilation and creation operators is given by

$$(a_\pm(f)\psi)^{(n)}(x_1, \ldots, x_n) = (n+1)^{1/2} \int dx\, \overline{f(x)}\psi^{(n+1)}(x, x_1, \ldots, x_n),$$

$$(a_\pm^*(f)\psi)^{(n)}(x_1, \ldots, x_n) = n^{-1/2} \sum_{i=1}^{n} (\pm 1)^{i-1} f(x_i)\psi^{(n-1)}(x_1, \ldots, \hat{x}_i, \ldots, x_n),$$

where $\hat{x}_i$ denotes that the ith variable is to be omitted. Note that as the maps

$$f \mapsto a_\pm(f), \qquad f \mapsto a_\pm^*(f)$$

are anti-linear and linear, respectively, one may introduce operator-valued distributions, i.e., fields, $a_\pm(x)$ and $a_\pm^*(x)$, such that

$$a_\pm(f) = \int dx\, \overline{f(x)} a_\pm(x), \qquad a_\pm^*(f) = \int dx\, f(x) a_\pm^*(x),$$

and then the action of these fields is given by

$$(a_\pm(x)\psi)^{(n)}(x_1, \ldots, x_n) = (n+1)^{1/2}\psi^{(n+1)}(x, x_1, \ldots, x_n),$$

$$(a_\pm^*(x)\psi)^{(n)}(x_1, \ldots, x_n) = n^{-1/2} \sum_{i=1}^{n} (\pm 1)^{i-1} \delta(x - x_i)\psi^{(n-1)}(x_1, \ldots, \hat{x}_i, \ldots, x_n).$$

In terms of these fields the number operator N is formally given by

$$N = \int dx\, a_\pm^*(x) a_\pm(x).$$

5.2.1.1. The CAR Relations. We next analyze the properties of the creation and annihilation operators obeying the CAR relations on the Fock space $\mathfrak{F}_-(\mathfrak{h})$. We simplify notation by dropping the suffix minus on the operators.

Proposition 5.2.2. *Let $\mathfrak{h}$ be a complex Hilbert space, $\mathfrak{F}_-(\mathfrak{h})$ the Fermi Fock space, and $a(f)$ and $a^*(g)$ the corresponding annihilation and creation operators on $\mathfrak{F}_-(\mathfrak{h})$. It follows that*

(1) $$\|a(f)\| = \|f\| = \|a^*(f)\|$$

for all $f \in \mathfrak{h}$, and hence $a(f)$ and $a^*(g)$ have bounded extensions.

(2) *If $\Omega = (1, 0, 0, \ldots)$ and $\{f_\alpha\}$ is an orthonormal basis of $\mathfrak{h}$, then*

$$\psi(f_{\alpha_1}, \ldots, f_{\alpha_n}) = a^*(f_{\alpha_1}) \cdots a^*(f_{\alpha_n})\Omega$$

is an orthonormal basis of $\mathfrak{F}_-(\mathfrak{h})$ when $\{f_{\alpha_1}, \ldots, f_{\alpha_n}\}$ runs over the finite subsets of $\{f_\alpha\}$.

(3) *The set of bounded operators*

$$\{a(f), a^*(g); f, g \in \mathfrak{h}\}$$

is irreducible on $\mathfrak{F}_-(\mathfrak{h})$.

PROOF. (1) One has

$$\begin{aligned}(a^*(f)a(f))^2 &= a^*(f)\{a(f), a^*(f)\}a(f) \\ &= \|f\|^2 a^*(f)a(f)\end{aligned}$$

and hence

$$\begin{aligned}\|a(f)\|^4 &= \|(a^*(f)a(f))^2\| \\ &= \|f\|^2\|a^*(f)a(f)\| = \|f\|^2\|a(f)\|^2.\end{aligned}$$

As $a(f) \neq 0$ for $f \neq 0$ one concludes that

$$\|a(f)\| = \|f\| = \|a^*(f)\|.$$

(2) This follows easily from the observation that

$$\psi(f_{\alpha_1}, \ldots, f_{\alpha_n}) = (n!)^{1/2} P_-(f_{\alpha_1} \otimes \cdots \otimes f_{\alpha_n}).$$

(3) Let T be a bounded operator in the commutant of $\{a(f), a^*(g); f, g \in \mathfrak{h}\}$, then

$$\begin{aligned}&(\psi(f_{\alpha_1}, \ldots, f_{\alpha_n}), T\psi(g_{\beta_1}, \ldots, g_{\beta_m})) \\ &\quad = (T^*\Omega, a(f_{\alpha_n}) \cdots a(f_{\alpha_1})a^*(g_{\beta_1}) \cdots a^*(g_{\beta_m})\Omega) \\ &\quad = (T^*\Omega, \Omega)(\psi(f_{\alpha_1}, \ldots, f_{\alpha_n}), \psi(g_{\beta_1}, \ldots, g_{\beta_m})).\end{aligned}$$

To establish the last equality one considers the three cases $n > m$, $n < m$, and $n = m$, separately. In the first case both expressions are zero because the $a(f)$ annihilate more particles than the $a^*(g)$ create. In the second case both expressions are again zero by complex conjugation. In the third case $a(f_{\alpha_n}) \cdots a^*(g_{\beta_m})\Omega$ is a multiple of Ω and the desired equality follows once more. Thus $T = (\Omega, T\Omega)\mathbb{1}$ and irreducibility is a consequence of Proposition 2.3.8.

5.2.1.2. The CCR Relations. The main qualitative difference between fermions and bosons is the absence of a Pauli principle for the latter particles. There is no bound on the number of particles which can occupy a given state. This is quantitatively reflected by the unboundedness of the Bose annihilation and creation operators. If, for example, $\psi^{(n)}$ is the n-fold tensor product of $f \in \mathfrak{h}$ with itself, then the annihilation operator satisfies

$$\|a(f)\psi^{(n)}\| = n^{1/2}\|\psi^{(n)}\|\,\|f\|$$

(we omit the suffix plus on the operators). This unboundedness leads to a large number of technical difficulties which are absent for fermions. These problems can be partially avoided by consideration of bounded functions of the operators $a(f)$ and $a^*(g)$.

It is convenient to introduce a family of operators $\{\Phi(f), f \in \mathfrak{h}\}$ by

$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}}.$$

Note that if $\Pi(f) = \Phi(if)$, then

$$\Pi(f) = \frac{a(f) - a^*(f)}{\sqrt{2}i}.$$

Thus

$$a(f) = \frac{\Phi(f) + i\Pi(f)}{\sqrt{2}}, \qquad a^*(f) = \frac{\Phi(f) - i\Pi(f)}{\sqrt{2}},$$

and the $a(f)$ and $a^*(f)$ can be recuperated from the $\Phi(f)$. Thus for functional purposes it suffices to examine the latter operators. Their basic properties are most easily examined on the subspace $F(\mathfrak{h}) \subset \mathfrak{F}_+(\mathfrak{h})$ formed by the finite-particle vectors, i.e., the $\psi = \{\psi^{(n)}\}_{n \geq 0}$ which have only a finite number of nonvanishing components.

Proposition 5.2.3. *Let $\mathfrak{h}$ be a complex Hilbert space, $\mathfrak{F}_+(\mathfrak{h})$ the Bose Fock space, and $a(f)$ and $a^*(g)$ annihilation and creation operators satisfying the canonical commutation relations. Define Φ by*

$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}}$$

for all $f \in \mathfrak{h}$. It follows that

(1) *For each $f \in \mathfrak{h}$, $\Phi(f)$ is essentially selfadjoint on $F(\mathfrak{h})$ and if $\|f_\alpha - f\| \to 0$, then $\|(\Phi(f_\alpha) - \Phi(f)\psi\| \to 0$ for all $\psi \in D(N^{1/2})$.*

(2) *If $\Omega = (1, 0, 0, \ldots)$ then the linear span of the set $\{\Phi(f_1)\cdots\Phi(f_n)\Omega;$ $f_i \in \mathfrak{h}, n = 0, 1, \ldots\}$ is dense in $\mathfrak{F}_+(\mathfrak{h})$.*

(3) *For each $\psi \in D(N)$ and $f, g \in \mathfrak{h}$ one has*

$$(\Phi(f)\Phi(g) - \Phi(g)\Phi(f))\psi = i\,\mathrm{Im}(f, g)\psi.$$