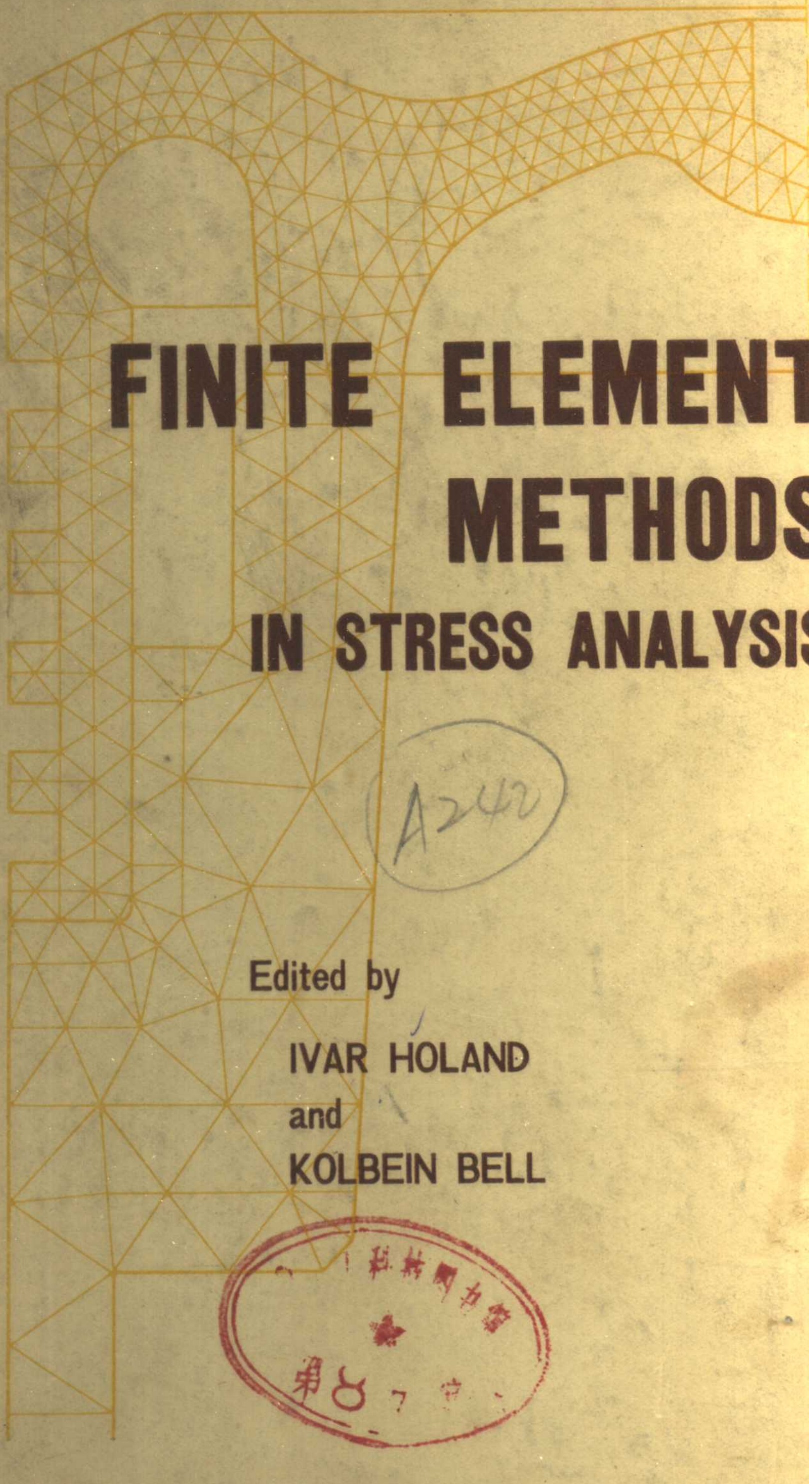

FINITE ELEMENT METHODS IN STRESS ANALYSIS

Edited by

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and

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Editors' Preface

This book is a result of a Scandinavian course on finite element methods arranged at The Technical University of Norway, January 6th to 11th, 1969, by The Division of Structural Mechanics in collaboration with The Norwegian Society of Professional Engineers. Apart from minor alterations and corrected printing errors this edition is based directly on a preliminary edition which was prepared to meet the need for written course material. The fact that the book is written in English reflects the editors' hope that it may be of interest to a wider, international circle of readers.

With the object of presenting a comprehensive and advanced course on finite element methods in stress analysis, including both theoretical aspects and practical applications, specialists associated with various Norwegian institutions actively engaged in this field were invited to contribute lectures fitting into the scope of the course. The institutions represented are:

The Norwegian Building Research Institute, Oslo.
Det norske Veritas (The Norwegian Ship Classification Society), Oslo.
The Technical University of Norway, Trondheim, represented by

SINTEF (The Engineering Research Foundation at The Technical University of Norway),
The Department of Ship Structures, and
The Division of Structural Mechanics.

Professor O.C. Zienkiewicz, University of Wales, Swansea, Great Britain, has added considerably to the value of the book by accepting an invitation to attend the course as guest lecturer invited under the auspices of the Foreign University Interchange Scheme, and by allowing his lecture to be included in the book (Chapter 13).

The editors have done their best to correlate the contents of the various chapters so as to prevent the book from being merely a collection of separate papers. In spite of these efforts a certain amount of overlapping and dislocation could not be avoided. Nevertheless, it is hoped that the book may serve as a textbook on the subject.

While most of the material included has been previously published elsewhere, some of the chapters contain new developments which are presented for the first time.

The editors would like to express their gratitude to the contributing authors for their willingness to fulfil the wishes of the editors regarding scope, notation and delivery time of the manuscripts.

During the preparation of the book Mrs. Irene Norvik, Mrs. Helle Wester and Mrs. Sigrun Belgum have never lost patience with the tedious job of typing the manuscripts, and their efforts are gratefully acknowledged.

Finally we would like to pay tribute to the publishers, Tapir Forlag, for their help and cooperation in preparing the material and for the way they kept to our very tight time schedule. Our gratitude is in particular due to Mr. Kjell Tangstad who was in charge of the production.

Trondheim, February 1969

Ivar Holand   Kolbein Bell

Notation

Attempts have been made to carry through a unified notation, and the following general rules have been adopted:

Matrix and vector symbols are represented by bold type characters throughout.

The transpose of a matrix or a vector is denoted by the superscript T.
It should be noted that a vector symbol without the superscript T is a column vector.

The partial derivative of for instance w with respect to x and y is written as

$$\frac{\partial^2 w}{\partial x \partial y} \quad \text{or alternatively as} \quad w_{,xy}$$

Numbers in special brackets ( ) refer to the list of references given at the end of each chapter.

Apart from Zienkiewicz et al. (Chapter 13) who use their own well-established symbols, the contributing authors were asked to use the following symbols:

SCALARS

A	area of plane elements
D	flexural rigidity of a plate
E	Young's modulus
G	shearing modulus
h	plate thickness
M_x, M_y, M_{xy}	moment components (plate bending)
Q_x, Q_y	shear force components (plate bending)
x,y,z	orthogonal Cartesian coordinates
u_x, u_y, u_z	displacement components (plane or three-dimensional stress problems)

V	volume of three-dimensional elements
w	deflection (plate bending)
ε, γ	strain components
σ, τ	stress components
ν	Poisson's ratio
ζ_i	area coordinates (i = 1,2,3), or volume coordinates (i = 1,2,3,4)

MATRICES AND VECTORS

$\mathbf{k}$	element stiffness matrix
$\mathbf{v}$	element nodal displacements (vector)
$\mathbf{S}$	element nodal forces (vector)
$\mathbf{S} = \mathbf{k}\mathbf{v}$	
$\mathbf{k}_q$	generalized element stiffness matrix
$\mathbf{q}$	generalized element nodal displacements (vector)
$\mathbf{Q}$	generalized element nodal forces (vector)
$\mathbf{Q} = \mathbf{k}_q\mathbf{q}$	
$\mathbf{K}$	structure stiffness matrix
$\mathbf{r}$	global nodal displacements (vector)
$\mathbf{R}$	global nodal forces (load vector)
$\mathbf{R} = \mathbf{K}\mathbf{r}$	
$\mathbf{I}$	identity matrix
$\mathbf{J}$	Jacobian matrix
$\mathbf{0}$	null matrix
$\mathbf{B}$ and $\mathbf{G}$	transformation matrices defined by the following relationships

$$\mathbf{Q} = \mathbf{G}\mathbf{S} \text{ and } \mathbf{S} = \mathbf{B}\mathbf{Q} \rightarrow \mathbf{B} = \mathbf{G}^{-1}$$

which imply

$$\mathbf{v} = \mathbf{G}^T\mathbf{q} \text{ and } \mathbf{q} = \mathbf{B}^T\mathbf{v}$$

Plane or three-dimensional problems:

$\mathbf{s}$	vector of stress components
$\mathbf{e}$	corresponding strain components
$\mathbf{E}$	elasticity matrix defined by the relation

$$\mathbf{s} = \mathbf{E}\mathbf{e}$$

Plate bending:

- $\mathbf{m}$ vector of moment components
- $\mathbf{c}$ corresponding curvature components
- $\mathbf{D}$ elasticity (stiffness) matrix defined by the relation

$$\mathbf{m} = -\mathbf{D}\mathbf{c}$$

Strain - displacement relations:

$$\mathbf{e} = \mathbf{P}\mathbf{v} \quad \text{or} \quad \mathbf{e} = \mathbf{P}_q\mathbf{q}$$

Curvature - displacement relations:

$$\mathbf{c} = \mathbf{P}\mathbf{v} \quad \text{or} \quad \mathbf{c} = \mathbf{P}_q\mathbf{q}$$

Other symbols used are defined where they first appear.

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INTRODUCTION

J. Moe

1 The Finite Element Technique – A New Tool in Structural Analysis

1.1 INTRODUCTION

The finite element technique is a relatively new and highly useful method for stress analysis of structural continua. The concept underlying this method was originally introduced by Argyris (1), who published a series of papers on this and related subjects in 1954-55, and by Turner, Clough, Martin and Topp in a paper published in 1956 (2). The method relies strongly on the matrix formulation of structural analysis which had been introduced a few years earlier by Langefors (3) and others, mainly as a result of the increasing use of electronic computers. Since 1956 there has been a concurrent and rapid development of electronic computers, matrix methods and finite element techniques. Hundreds of papers are now published every year on matrix methods and finite element techniques. Several important conferences have been devoted exclusively to these subjects (4,19), and the first textbooks have recently appeared (5,6). Any stress analyst who has been actively engaged in his field for more than ten years has experienced how the new techniques have completely changed the scope of problems which are amenable to numerical analysis. In this introductory chapter the concept of the finite element technique will be discussed along with the inherent advantages and disadvantages. Some examples of practical applications are presented. The mathematical treatment as well as numerous additional examples are presented in the following chapters.

1.2 THE FINITE ELEMENT CONCEPT

Two- and three-dimensional bodies such as plates, shells, dams etc. may be thought of as internally statically indeterminate structures for which the degree of indeterminacy is infinite. Stress and displacement fields of certain special classes of such structures are available in terms of closed solutions to sets of governing differential equations. Unfortunately the majority of practical design problems fall outside the reach of such closed solutions, due to complex and irregular geometric forms of the structures, complexity in loading patterns, non-linearity and inhomogeneity in properties of materials etc. In such cases the designer must resort to approximate solutions. The traditional approach is then to establish a simplified and idealized model of the real structure, for which a closed solution is available, and apply this solution with proper interpretation. Other approaches also exist, such as the finite difference technique and model experiments.

The first step in the finite element technique may also be thought of as a modelling of the real structure into a simplified and idealized system for which a solution is available. There is, however, as will be shown, a very important difference in the flexibility of modelling and hence the accuracy with which the idealized system can be brought to match the real structure.

The approach is most easily appreciated when discussed in relation to a simple example. Consider, therefore, the bracket shown in Fig. 1.1a, which is loaded as indicated. Stresses and displacements are sought. The analysis is performed on the idealized model shown in Fig. 1.1b. This model consists of a number of elements obtained by means of fictitious cuts through the original structure, as indicated by the lines in Fig. 1.1b. Adjoining elements may be thought of as being connected at common points (termed "nodes" or "nodal points"), but are elsewhere separated by the imaginary cuts. In Fig. 1.1b the elements are shown as triangles. They could have other forms, too.

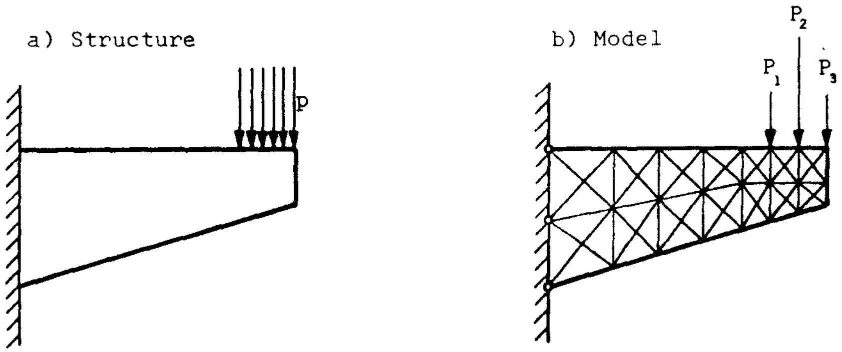

FIG. 1.1 Actual structure and finite element model

The idealized model is now analysed in a manner which is in principle identical to the well known displacement method (or slope-deflection method) of frame analysis. When the structure is deformed due to external loads or initial strains, it is required that continuity between neighbouring elements must be maintained at the common nodes. The displacements of the nodes are introduced as the primary indeterminate quantities. Nodal point forces are expressed in terms of nodal point displacements and the nodal point displacements are determined by means of the equilibrium conditions at the nodes. The degree of kinematic indeterminacy (i) is equal to the number of unknown independent displacement components at the nodes:

$$i = k \cdot n - \ell$$

where:

n = total number of nodes

k = number of independent displacement components (degrees of freedom) at each node

ℓ = number of prescribed nodal point displacements at the boundaries.