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Jens M. Melenk

hp-Finite Element Methods for Singular Perturbations

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Preface

Many partial differential equations arising in practice are parameter-dependent problems and are of singularly perturbed type for small values of this parameter. These include various plate and shell models for small thickness in solid mechanics, the convection-diffusion equation, Oseen equation, and Navier-Stokes equations in fluid flow problems where the fluid is assumed to have small viscosity, and finally equations arising in semi-conductor device modelling. Analysis of such equations by numerical methods such as the finite element method is an important task in today's computational practice. A significant design aspect of numerical methods for such parameter-dependent problems is robustness, that is, that the performance of the numerical method is independent of, or at least fairly insensitive to, the parameter. Numerous methods have been proposed and analyzed both theoretically and computationally for a variety of singularly perturbed problems—we merely refer at this point to the three recent monographs [97, 99, 108] and their extensive bibliographies.

Most numerical methods employed in the study of singularly perturbed problems are low order methods such as the classical h-version finite element method (FEM), where convergence is obtained by refining the mesh while keeping the approximation order fixed. In high order methods such as the hp-version of the finite element method (hp-FEM), mesh refinement can be combined with increasing the approximation order; for certain problem classes, this added flexibility of the hp-FEM allows it to achieve exponential rates of convergence.

The present book is devoted to a complete analysis of the hp-FEM for a class of singularly perturbed problems on curvilinear polygons. To the knowledge of the author this work represents the first *robust exponential convergence* result for a class of singularly perturbed problems under realistic assumptions on the input data, that is, piecewise analyticity of the coefficients of the differential equation and the geometry of the domain.

This work is at the intersection of several active research areas that have their own distinct approaches and techniques: numerical methods for singular perturbation problems, high order numerical methods for elliptic problems in non-smooth domains, regularity theory for singularly perturbed problems in terms of asymptotic expansions, and regularity theory for elliptic problems in curvilinear-polygons. Although, naturally, the present work draws on techniques employed in all of these fields, new tools and regularity results for the solutions had to be developed for a rigorous robust exponential convergence proof.

This book comprises research undertaken during my years at ETH Zürich. I take this opportunity to thank Prof. Dr. C. Schwab for many stimulating discussions on the topics of this book and for his support and encouragement over the years.

Leipzig, June 2002 *J.M. Melenk*

Notation

General notation

$\mathbb{N}$	The set of positive integers, $\{1, 2, \dots, \}$.
$\mathbb{N}_0$	The set of non-negative integers $\mathbb{N} \cup \{0\}$.
$\mathbb{Z}$	The set of integers $\mathbb{N}_0 \cup -\mathbb{N}$.
$\mathbb{R}, \mathbb{R}^+, \mathbb{R}_0^+$	The real, the positive real, and the non-negative real numbers.
$\mathbb{C}$	The complex numbers.
$\mathbf{i}$	The imaginary unit with $\mathbf{i}^2 = -1$.
$\Gamma(\cdot)$	The Gamma function with $\Gamma(j+1) = j!$ for $j \in \mathbb{N}_0$.
δ_{ij}	The Kronecker symbol: $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$.
C, C', γ, γ', K, b	Generic constants independent of critical parameters such as ε, the differentiation order, the polynomial degree, etc. These constants may be different in different instances.
$\lfloor \cdot \rfloor$	$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$.
$[\cdot]$	In Section 5.5: $[p] = \max\{1, p\}$ for $p \in \mathbb{Z}$ (see p. 198). In all other sections $[\cdot]$ denotes the jump operator.
$\subset\subset$	Compact embedding.
$\lvert K \rvert$	for $K \subset \mathbb{R}^n$ represents the Lebesgue measure (volume) of K.
$\mathbb{E}_\Omega$	The characteristic function of the set Ω.

Matrices

$\mathbb{M}^n$	The set of (real) $n \times n$ matrices.
$\mathbb{S}^n$	$\mathbb{S}^n \subset \mathbb{M}^n$ are the symmetric matrices.
$\mathbb{S}^n_>$	The set of symmetric positive definite matrices.
$:$	For matrices A, B, we set $A : B = \sum_{i,j} A_{ij} B_{ij}$.

Sets, balls, sectors, neighborhoods

$B_r(x)$	The ball of radius r around the point x.
B_R, B_R^+, B_R^-	Ball and half balls with radius R; see (5.5.1).
$\mathcal{U}_\kappa(K)$	The κ-neighborhood of the set K, i.e., $\cup_{x \in K} B_\kappa(x)$.
S	A generic sector, Definition 4.2.1, p. 146.
$S_R(\omega)$	A sector with opening angle ω, see (4.1.2).

$S_R^{0,\delta}(\omega)$, $S_R^{\omega,\delta}(\omega)$	Conical neighborhood of the lateral parts Γ_0, Γ_ω of the sector $S_R(\omega)$, see (5.4.30).
I_x, I_y	Intervals on $\mathbb{R}$; I_y is the form $I_y = [0, b]$ for a $b > 0$; see outset of Section 7.3.1.
S_X, $S_X(\delta)$	Complex neighborhoods of interval I_x, see (7.3.11).

Norms, differential operators, standard function spaces

$L^2(\Omega)$	The space of square integrable functions.
$H^k(\Omega)$	Sobolev space H^k of L^2-functions whose distributional derivatives of order up to k are also in L^2; cf. [1].
$H_0^1(\Omega)$	Sobolev space of H^1-functions with vanishing trace on $\partial\Omega$; cf. [1].
$H_{0,\varepsilon}^1(\Omega)$	The Sobolev space of H^1 functions with vanishing trace on $\partial\Omega$ equipped with the energy norm $\|\cdot\|_{L^2(\Omega)} + \varepsilon\|\nabla\cdot\|_{L^2(\Omega)}$; cf. p. 184.
$H^{1/2}(\Omega)$	The usual Sobolev space $H^{1/2}$; cf. [1].
$H_{00}^{1/2}(\Omega)$	The usual Sobolev space $H_{00}^{1/2}$; cf. [1].
$\|\cdot\|_\varepsilon$	Energy norm $\|u\|_\varepsilon \sim \|u\|_{L^2(\Omega)} + \varepsilon\|\nabla u\|_{L^2(\Omega)}$; cf. p. 3.
$\|\cdot\|_{\varepsilon,\alpha}$	Exponentially weighted energy norm, cf. p. 243.
$D^\alpha u$	For multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and (smooth) functions u defined on an open subset of $\mathbb{R}^n$: $D^\alpha u = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n} u$.
$\nabla^p u(x)$	$\lvert\nabla^p u(x)\rvert^2 = \sum_{\alpha_1,\ldots,\alpha_p=1}^{n} \lvert\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_p} u(x)\rvert^2$, where, for tensor-valued functions $u = (u_i)_{i=1}^N$ and shorthand $\partial_\alpha u$ for $\partial_{x_\alpha} u$ $\lvert\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_p} u(x)\rvert^2 = \sum_{i=1}^{N} \lvert\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_p} u_i(x)\rvert^2$.
$\mathcal{A}(G)$	For domains $G \subset \mathbb{R}^n$ (or $\mathbb{C}^n$) $\mathcal{A}(G)$ denotes the set of functions analytic on G. For closed sets $\overline{G}$, $f \in \mathcal{A}(\overline{G})$ is understood to imply the existence of an open neighborhood of $\overline{G}$ on which f is analytic; see also (1.2.2).
$\mathcal{A}(G, \mathbb{R}^n)$	The set of vector-valued functions that are (componentwise) analytic on G.
$\mathcal{A}(G, \mathbb{S}_>^n)$	The set of functions from G to the symmetric positive definite matrices $\mathbb{S}_>^n$ that are (componentwise) analytic on G.
L_ε	The differential operator, (1.2.1).
$[\cdot]$	The jump operator across a curve. Only in Sec. 5.5: $[p] = \max\{1, p\}$ for $p \in \mathbb{Z}$; p. 198.
∂_{n_A}	The co-normal derivative operator $\mathbf{n}^T\nabla\cdot$.

Weight functions and weighted spaces

$\hat{\Phi}_{p,\beta,\varepsilon}$	Weight function in a sector, p. 147.
$\Phi_{p,\beta,\varepsilon}$	Weight function in a curvilinear polygon, p. 184.
$\hat{\Psi}_{p,\beta,\varepsilon,\alpha}$	Exponentially weighted weight function in a sector, p. 231.
$\Psi_{p,\beta,\varepsilon,\alpha}$	Exponentially weighted weight function in Ω; see (7.4.7).
$H^{m,l}_{\beta,\varepsilon}$	Weighted Sobolev space, p. 149.
$\mathcal{B}^{l}_{\beta,\varepsilon}$	Countably normed space, p. 149.
$H^{m,l}_{\beta,\varepsilon,\alpha}$	Exponentially weighted Sobolev space, p. 232.
$\mathcal{B}^{l}_{\beta,\varepsilon,\alpha}$	Exponentially weighted countably normed space, p. 232.
$\mathcal{E}$	Smallest characteristic length scale, p. 177.

Semi norms for controlling high order derivatives

$N_{R,p}(u)$, $N'_{R,p,q}(u)$, $N'^{,\pm}_{R,p,q}(u)$	Bounds on higher derivatives of u, p. 198.
$M_{R,p}(f)$	Bounds on higher derivatives of f, p. 202.
$M'_{R,p}(f)$, $\tilde{M}_{R,p}(f)$, $N'_{R,p}(u)$	Bounds on higher derivatives of f and u, p. 208.
$N'^{,\pm}_{R,p}(u)$, $H_{R,p}(u)$, $M'^{,\pm}_{R,p}(f)$	Bounds on higher derivatives of u and f, p. 217.

Description of the boundary and corner layer

ψ_j	Boundary fitted coordinates $(x,y) = \psi_j(\rho_j, \theta_j)$ in neighborhood of arc Γ_j, where ρ_j measures the distance of the point (x,y) to Γ_j; see Notation 2.3.3
A_j	Vertex of the curvilinear domain Ω, Section 1.2.
Γ_j	Analytic arc being part of the boundary of the curvilinear polygon Ω, Section 1.2.
S_j, S_j^+, S_j^-	Sectors near A_j for the definition of corner layer, (7.4.2), (7.4.3).
Ω_j, χ^{BL}, χ^{CL}	Subdomains of Ω and cut-off functions associated with arcs Γ_j and the vertices A_j; see Notation 2.3.3 and outset of Section 7.4.1.
Ω_j, χ_j^{BL}, χ_j^{CL}	Subdomains of Ω and cut-off functions associated with arcs Γ_j and the vertices A_j; see Notation 2.3.3.
s_κ, $\tilde{s}_{j,\kappa}$	Anisotropic and anisotropic stretching maps; see Notation 2.4.3.

Polynomials, approximation, and projections

I, S, T	The reference interval $I = (0,1)$, square $S = I \times I$, and triangle $T = \{(x,y) \,\vert\, 0 < x < 1, 0 < y < 1 - x\}$, p. 87.
$\underline{S}$, $\underline{T}$	The references square and triangle in Section 3.2.3; see (3.2.19).
$\mathcal{P}_p(T)$, $\mathcal{Q}_p(S)$, $\Pi_p(K)$	Spaces of polynomials, p. 87.
$P_p^{(\alpha,\beta)}$	Jacobi polynomials, see [124].
L_p, $\tilde{L}_p$	$L_p = P_p^{(0,0)}$ is the usual Legendre polynomial; $\tilde{L}_p(x) = L_p(2x-1)$.
$\psi_{p,q}$	Orthogonal polynomials on the triangle, (3.2.23).
$\mathcal{GL}$	Gauss-Lobatto points, p. 87.
i_p, j_p	1D and 2D Gauss-Lobatto interpolation operators, p. 87.
$i_{p,\Gamma}$	Gauss-Lobatto interpolation operator on an edge Γ, p. 88.
E	Polynomial extension operator from the boundary, p. 89.
Π_p^{∞}	Polynomial projector defined in Theorem 3.2.20, p. 103.
$\Pi_p^{1,\infty}$	Polynomial projector defined in Theorem 3.2.24, p. 108.
$\Pi_p^{L^2}$	The L^2 projector into the space $\mathcal{P}_p$.

Meshes and finite element approximation

$\mathcal{T}$	Triangulation, p. 39.
$S^p(\mathcal{T})$, $S_0^p(\mathcal{T})$	Spaces of piecewise mapped polynomials of degree p on the mesh $\mathcal{T}$, p. 113.
$\Pi_{p,\mathcal{T}}^{\infty}$	Elementwise application of Π_p^{∞} on a mesh $\mathcal{T}$, p. 113.

Contents

1. Introduction

1.1 Introduction

1.2 Problem class and assumptions

This work presents numerical analysis and regularity results for singularly perturbed equations of the form (1.2.1). Such equations are ubiquitous, appearing, for example in convection-dominated fluid flow, in semi-conductor device modelling, and solid mechanics (where their analysis is crucial for an understanding of the layer structure of Reissner-Mindlin plate models, [9, 10]).
We consider the following class of singularly perturbed equations:

$$L_\varepsilon u_\varepsilon := -\varepsilon^2 \nabla \cdot (A(x) \nabla u_\varepsilon) + b(x) \cdot \nabla u_\varepsilon + c(x) u_\varepsilon = f \text{ on } \Omega, \quad (1.2.1a)$$

$$u_\varepsilon = g \text{ on } \partial\Omega. \quad (1.2.1b)$$

The bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ is assumed to be a *curvilinear polygon* as depicted in Fig. 1.2.1. The boundary $\partial\Omega$ is assumed to consist of finitely

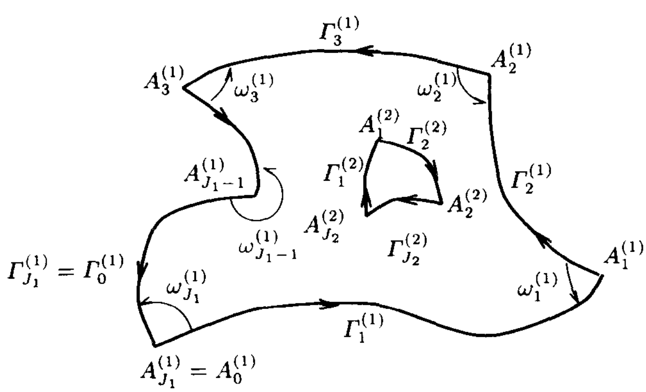

Fig. 1.2.1. A curvilinear polygon.

many curves $\Gamma^{(i)}$, i.e., $\partial\Omega = \cup_{i=1}^N \Gamma^{(i)}$, each of which consists of finitely many *analytic arcs* $\Gamma_j^{(i)}$:

$$\Gamma^{(i)} = \cup_{i=1}^{J_i} \overline{\Gamma_j^{(i)}}.$$

The arcs $\Gamma_j^{(i)}$ are parametrized by

$$\Gamma_j^{(i)} = \{(x_j^{(i)}(\theta), y_j^{(i)}(\theta) \,|\, \theta \in (0,1)\},$$

where the functions $x_J^{(i)}$, $y_j^{(i)}$ are analytic on a neighborhood of the interval $[0,1]$. We assume that

$$\left|\frac{d}{d\theta}x_j^{(i)}\right|^2 + \left|\frac{d}{d\theta}y_j^{(i)}\right|^2 > 0 \quad \text{on } [0,1] \text{ for all } i,\ j.$$

The curves $\Gamma_j^{(i)}$ are oriented such that the domain Ω is "on the left"; that is, the normal vector $(-\frac{d}{\theta}y_j^{(i)}(\theta), \frac{d}{\theta}x_j^{(i)}(\theta))$ points into Ω (cf. Fig. 1.2.1). The endpoints of the arc $\Gamma_j^{(i)}$ are the vertices $A_{j-1}^{(i)} = (x_j^{(i)}(0), y_j^{(i)}(0))$, $A_j^{(i)} = (x_j^{(i)}(1), y_j^{(i)}(1))$, and we set $A_0^{(i)} := A_{J_i}^{(i)}$. The internal angle at vertex $A_j^{(i)}$ is denoted $\omega_j^{(i)}$, and we exclude cusps by stipulating $0 < \omega_j^{(i)} < 2\pi$. In order to simplify the notation in this work, we assume without loss of generality that $N = 1$ and drop the superscript (i); i.e., we write $J = J_1$, $\Gamma_j = \Gamma_j^{(1)}$, $A_j = A_j^{(1)}$, etc. It is also convenient to write $\Gamma_0 = \Gamma_J$.

The remaining data appearing in (1.2.1) are assumed to be analytic: We suppose that $c \in \mathcal{A}(\overline{\Omega})$, $b \in \mathcal{A}(\overline{\Omega}, \mathbb{R}^2)$, and $A \in \mathcal{A}(\overline{\Omega}, \mathbb{S}^2_>)$; i.e., we stipulate the existence of C_A, C_b, C_c, and γ_A, γ_b, $\gamma_c > 0$ such that

$$\|\nabla^p A\|_{L^\infty(\Omega)} \le C_A \gamma_A^p p! \qquad \forall p \in \mathbb{N}_0, \tag{1.2.2a}$$

$$\|\nabla^p b\|_{L^\infty(\Omega)} \le C_b \gamma_b^p p! \qquad \forall p \in \mathbb{N}_0, \tag{1.2.2b}$$

$$\|\nabla^p c\|_{L^\infty(\Omega)} \le C_c \gamma_c^p p! \qquad \forall p \in \mathbb{N}_0. \tag{1.2.2c}$$

Furthermore, the matrix $A(x)$ is symmetric positive definite for each $x \in \Omega$ and there exists $\lambda_{min} > 0$ such that

$$A(x) \ge \lambda_{min} \qquad \forall x \in \Omega. \tag{1.2.2d}$$

We require the existence of $\mu > 0$ such that

$$-\frac{1}{2}(\nabla \cdot b)(x) + c(x) \ge \mu > 0 \qquad \forall x \in \Omega. \tag{1.2.2e}$$

The right-hand side f in (1.2.1) satisfies $f \in \mathcal{A}(\overline{\Omega})$, i.e., there are C_f, $\gamma_f > 0$ such that

$$\|\nabla^p f\|_{L^\infty(\Omega)} \le C_f \gamma_f^p p! \qquad \forall p \in \mathbb{N}_0. \tag{1.2.3}$$

Finally, the boundary data $g \in C(\partial\Omega)$ are assumed to be analytic on the arcs Γ_j: For each j, the function $g(x_j, y_j)$ is analytic on $[0,1]$, i.e., there are C_g, $\gamma_g > 0$ such that

$$\|D^p g(x_j(\cdot), y_j(\cdot))\|_{L^\infty((0,1))} \le C_g \gamma_g^p p! \qquad \forall p \in \mathbb{N}_0. \tag{1.2.4}$$

For most of our analysis, the singular perturbation parameter $\varepsilon \in (0,1]$ is assumed to be small, i.e., $\varepsilon \ll 1$.

Remark 1.2.1 The assumption of analyticity of the data on Ω can be relaxed. In fact, in most of the subsequent analysis, only piecewise analyticity of the data A, b, c, f, and g needs to be assumed. ▪

Solutions of (1.2.1) are understood in the weak sense; i.e., u_ε is the solution of the following problem:

$$\text{Find } u_\varepsilon \in H^1(\Omega) \text{ s.t. } u_\varepsilon|_{\partial\Omega} = g \text{ and } B_\varepsilon(u_\varepsilon, v) = F(v) \qquad \forall v \in H_0^1(\Omega). \quad (1.2.5)$$

Here, the bilinear form B_ε and the linear form F are defined by

$$B_\varepsilon(u,v) := \varepsilon^2 \int_\Omega (A(x)\nabla u)\cdot\nabla v + b(x)\cdot\nabla u v + c(x)uv\,dx, \qquad (1.2.6a)$$

$$F(v) := \int_\Omega f(x)v\,dx. \qquad (1.2.6b)$$

This bilinear form B_ε is closely connected with the *energy norm*

$$\|u\|_\varepsilon^2 := \varepsilon^2\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \qquad (1.2.7)$$

as will become apparent in the subsequent Lemma 1.2.2. The bilinear form B_ε is coercive on the space $H_0^1(\Omega)$, and the variational formulation (1.2.5) has a unique solution even under the weaker assumptions $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$:

Lemma 1.2.2. *Let the coefficients A, b, c satisfy (1.2.2), $\varepsilon \in (0,1]$, and let $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$. Then there exists a unique solution u_ε of (1.2.5) and a constant $C > 0$ independent of ε, f, and g such that*

$$\varepsilon\|\nabla u_\varepsilon\|_{L^2(\Omega)} + \|u_\varepsilon\|_{L^2(\Omega)} \le C\left[\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Omega)}\right].$$

Moreover, the bilinear form B_ε is coercive on the space $H_0^1(\Omega)$, and there holds

$$B_\varepsilon(u,u) \ge \varepsilon^2\lambda_{min}\|\nabla u\|^2_{L^2(\Omega)} + \mu\|u\|^2_{L^2(\Omega)} \qquad \forall u \in H_0^1(\Omega). \qquad (1.2.8)$$

The bilinear form B_ε is also continuous on the space $H^1(\Omega)$: There is $C > 0$ independent of A, b, c, and ε such that for all u, $v \in H^1(\Omega)$ we have:

$$|B_\varepsilon(u,v)| \le C\left[\|A\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} + \|b\|_{L^\infty(\Omega)}\varepsilon^{-1}\right]\|u\|_\varepsilon\,\|v\|_\varepsilon. \quad (1.2.9)$$

Proof: (1.2.9) follows immediately from the Cauchy-Schwarz inequality.
As a first step, we show (1.2.8). We start by noting that for $u \in H_0^1(\Omega)$, an integration by parts gives

$$\int_\Omega (b\cdot\nabla u)u\,dx = -\int_\Omega (\nabla\cdot b)u^2 + u(b\cdot\nabla u)\,dx.$$

Therefore,

$$\int_\Omega (b\cdot\nabla u)u\,dx = -\frac{1}{2}\int_\Omega (\nabla\cdot b)u^2\,dx.$$

Combining this with assumption (1.2.2e) implies the coercivity of the bilinear form B_ε on the space $H_0^1(\Omega)$:

$$
\begin{aligned}
B_\varepsilon(u,u) &= \varepsilon^2 \int_\Omega (A(x)\nabla u)\cdot\nabla u + (b(x)\cdot\nabla u)u + c(x)u^2\,dx \\
&= \varepsilon^2 \int_\Omega (A(x)\nabla u)\cdot\nabla u + \left(c(x) - \frac{1}{2}\nabla\cdot b(x)\right)u^2\,dx \\
&\geq \varepsilon^2\lambda_{min}\|\nabla u\|^2_{L^2(\Omega)} + \mu\|u\|^2_{L^2(\Omega)}.
\end{aligned}
$$

This coercivity gives uniqueness of the solution of (1.2.5). In order to see existence of a solution, let $G \in H^1(\Omega)$ be an extension of g into Ω satisfying

$$
G|_{\partial\Omega} = g, \qquad \|G\|_{H^1(\Omega)} \leq C\|g\|_{H^{1/2}(\partial\Omega)}
$$

for some $C > 0$ depending only on Ω. The difference $\tilde{u} := u_\varepsilon - G$ must be the solution of the problem:

$$
\text{Find } \tilde{u} \in H_0^1(\Omega) \text{ s.t. } B_\varepsilon(\tilde{u},v) = F(v) - B_\varepsilon(G,v) \qquad \forall v \in H_0^1(\Omega). \tag{1.2.10}
$$

We see that for all $v \in H^1(\Omega)$

$$
\begin{aligned}
&|F(v) - B_\varepsilon(G,v)| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\quad + C\left\{\|G\|_{H^1(\Omega)}\varepsilon^2\|\nabla v\|_{L^2(\Omega)} + \|G\|_{H^1(\Omega)}\|v\|_{L^2(\Omega)} + \|G\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\right\} \\
&\leq C\left\{\|f\|_{L^2(\Omega)} + \|G\|_{H^1(\Omega)}\right\}\|v\|_\varepsilon,
\end{aligned}
$$

where we assumed $\varepsilon \leq 1$. Therefore, by the classical Lax-Milgram Lemma, [36, 82], (1.2.10) indeed has a unique solution $\tilde{u}$ satisfying

$$
\|\tilde{u}\|_\varepsilon \leq C\left[\|f\|_{L^2(\Omega)} + \|G\|_{H^1(\Omega)}\right].
$$

Using $\varepsilon \leq 1$, we see that $u_\varepsilon := G + \tilde{u}$ satisfies the desired bounds. □

The greater part of our analysis will be done for the special case $b \equiv 0$; i.e., we consider the following singularly perturbed problem of elliptic-elliptic type:

$$
\begin{aligned}
-\varepsilon^2\nabla\cdot(A(x)\nabla u_\varepsilon) + c(x)u_\varepsilon &= f \qquad \text{on } \Omega, && (1.2.11a)\\
u_\varepsilon &= g \qquad \text{on } \partial\Omega, && (1.2.11b)
\end{aligned}
$$

where assumption (1.2.2e) implies that $c \geq \mu > 0$ on Ω.

1.3 Principal results

The main result of the present work is the *robust exponential convergence* result Theorem 2.4.8 for high order finite element methods applied to (1.2.11). It is shown that with the proper choice of conforming subspaces V_N of dimension

$N \in \mathbb{N}$, the finite element method, i.e., Galerkin projection, yields approximants u_ε^N to the exact solution u_ε that satisfy

$$\|u_\varepsilon - u_\varepsilon^N\|_\varepsilon \le Ce^{-bN^{1/3}}. \tag{1.3.1}$$

Here, the constants C, $b > 0$ are independent of ε; in fact, in our numerical experiments in Section 2.5 $b \approx 1$ and likewise $C = O(1)$. The finite element spaces V_N are given explicitly in Section 2.4. They consist of the usual piecewise polynomial spaces of degree p defined on meshes that are adapted to the length scale ε of the problem. Specifically, for the approximation with polynomial of degree p, these meshes are designed according to three principles:

1. near the edges of the domain, long, thin *needle* elements of width $O(p\varepsilon)$ are employed in order to capture boundary layer phenomena;
2. in an $O(p\varepsilon)$ neighborhood of the vertices a *geometric* mesh refinement is used in order to resolve corner singularities;
3. in the interior of the domain a standard coarse mesh is utilized for the resolution of smooth solution components.

It is worth stressing that the only information required for an application of these mesh design principles is the length scale ε of the problem, which is typically known in practice.

Let us compare our robust exponential convergence result with previous convergence analyses. Thus far only algebraic robust convergence results have been available, typical of low order methods. Robust algebraically convergent methods deliver approximate solutions u_ε^N from spaces V_N of dimension $N \in \mathbb{N}$ that satisfy error bounds of the form

$$\|u_\varepsilon - u_\varepsilon^N\|_\varepsilon \le CN^{-\alpha}. \tag{1.3.2}$$

Here, C, $\alpha > 0$ are independent of ε. Even for optimally chosen meshes, $\alpha \le 2$ is typical for two-dimensional problems. A good measure for comparing approximation results (1.3.1) and (1.3.2) is the alphanumerical work W required to compute the approximate solution u_ε^N. In the case of low order methods, an efficient iterative solver such as multigrid, [65], is essential for acceptable solution times. Such an optimal solution algorithm would solve the resulting linear system with linear complexity, i.e., $W = O(N)$. The best rate of convergence of these low order methods in terms of work W is therefore

$$\|u_\varepsilon - u_\varepsilon^N\|_\varepsilon \le CW^{-\alpha}.$$

This work estimate, however, is based upon two strong assumptions. First, in order for α to be reasonable, e.g., $\alpha \approx 1$, the mesh has to be carefully designed so as to capture the relevant features of the solution. In particular, it has to contain highly anisotropic elements in the boundary layer. However, most state-of-the-art adaptive strategies do not allow for such elements: Their use of shape-regular elements precludes robustness, and the convergence rates visible in practice are