Advances in

HEAT TRANSFER

Edited by

James P. Hartnett Thomas F. Irvine, Jr.

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James P. Hartnett

Department of Energy Engineering
University of Illinois
at Chicago
Chicago, Illinois

Thomas F. Irvine, Jr.

State University of New York
at Stony Brook
Stony Brook, Long Island
New York

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LIST OF CONTRIBUTORS

R. D. CESS, *College of Engineering, State University of New York, Stony Brook, New York*

I. J. KUMAR, *Defense Science Laboratory, Delhi-6, India*

SIMON OSTRACH, *Division of Fluid, Thermal, and Aerospace Sciences, Case Western Reserve University, Cleveland, Ohio*

S. N. TIWARI,* *College of Engineering, State University of New York, Stony Brook, New York*

Z. ZARIĆ, *Boris Kidrič Institute, University of Beograd, Beograd, Yugoslavia*

A. ŽUKAUSKAS, *Academy of Sciences of the Lithuanian SSR, Vilnius, USSR*

* Present address: Department of Thermal Engineering, College of Engineering, Old Dominion University, Norfolk, Virginia 23508.

PREFACE

The serial publication "Advances in Heat Transfer" is designed to fill the information gap between the regularly scheduled journals and university level textbooks. The general purpose of this series is to present review articles or monographs on special topics of current interest. Each article starts from widely understood principles and in a logical fashion brings the reader up to the forefront of the topic. The favorable response to the volumes published to date by the international scientific and engineering community is an indication of how successful our authors have been in fulfilling this purpose.

The editors are pleased to announce the publication of Volume 8 and wish to express their appreciation to the current authors who have so effectively maintained the spirit of the series.

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Z. ZARIĆ

Recent Mathematical Methods in Heat Transfer

I. J. KUMAR

Defense Science Laboratory, Delhi-6, India

I. Introduction

With the recent advancements in the fields of atomic energy, aeronautics, and astronautics, an engineer is faced with more and more complex problems in heat transfer. While sophisticated instrumentation has greatly helped him to achieve accuracy and reliability in experimental measurements, the computer has immensely increased his capacity to theoretically study more realistic models. These advancements, however, are no substitute for the power and ingenuity of the mathematical methods which are, and would remain as, the main tool in the hands of engineers for the solution of practical problems. In the present review some of the most important methods employed in recent heat transfer literature will be reviewed, with a special stress on those still under development, and examples from the literature cited, to demonstrate the power of a method and indicate the branches of heat transfer where one may look upon a particular method as a potential tool for obtaining the solution. While reviewing a particular method, examples will be presented from the recent literature irrespective of the location of the problem in the hierarchy of heat transfer literature. Thus, it will be attempted to synthesize the recent developments in heat transfer from the point of view of mathematical methods. From a physical point of view, these developments have already been brought to light in a systematic and cohesive manner in the recently published monographs [1–7]. No claim to completeness of the review is made in view of the limitations of space and the availability of other literature, especially of Soviet literature.

No reference will be made in this review to the method of weighted residuals or integral methods. A review of these methods by Finlayson and Scriven [8] and an account of their application to unsteady heat conduction [9] have recently appeared. Similarly, the classical method of operational calculus used extensively by Carslaw and Jaeger [10] and Luikov [11] in the theory of heat conduction and by Luikov and Mikhaylov [12] in the theory of combined heat and mass diffusion will not be touched upon.

It would not be out of place here to mention a few words about nomenclature. Although highly desirable, it is very difficult to achieve uniform nomenclature in a review article of this nature, where various branches of heat transfer and many mathematical methods used therein are being reviewed. Therefore, for ease of reference, a comprehensive nomenclature has been provided separately for each section at the end of the review. The symbols which have the same meaning throughout the review have been listed under the section where they were used for the first time.

II. Perturbation Methods

The perturbation method consists essentially of expanding the dependent variable in a series of powers of a quantity known to be small. When this small quantity is a parameter the method is known as parameter perturbation and where it is a coordinate the method is termed a coordinate perturbation. Taking this small quantity to be ϵ, the solution of the differential equation for ϵ is the zeroth-order solution or the solution of the unperturbed problem. When the expansion is substituted in the differential equation and the like powers of ϵ equated, we get a system of differential equations for the subsequent order solutions. The assumed series is convergent in the asymptotic sense [13] and if the above scheme succeeds, we speak of it as a regular perturbation. This method has been used in a number of problems and has produced very useful results.

In many problems, however, the ratio of successive terms in the solution ceases to be small and the regular perturbation scheme therefore fails in some region of the flow field. Thus it is not possible to obtain a uniformly valid solution throughout the region of interest by a regular perturbation scheme. Such problems are known as singular perturbation problems. We propose to discuss some of the methods of dealing with such problems in the context of heat transfer.

A. PLK METHOD

Sometimes the regular perturbation fails because of the presence of a singularity in the zeroth-order solution at a point or on a line in the region of investigation. This singularity becomes progressively more and more severe as the order of the solution increases. A technique for solving such problems by a perturbation method was presented by Lighthill [14], in which the dependent variable v and the independent variable x are both expanded in powers of the small quantity ϵ. The method consists of expanding v and x thus:

$$v = v_0(\chi) + \epsilon v_1(\chi) + \epsilon^2 v_2(\chi) + \cdots, \tag{1}$$

$$x = \chi + \epsilon x_1(\chi) + \epsilon^2 x_2(\chi) + \cdots, \tag{2}$$

where χ takes the place of the original independent variable x, $v_0(\chi)$ is simply the zeroth-order solution of the regular perturbation method with χ replacing x, and $x_n(\chi)$, $n = 1, 2, 3, \ldots$ are so chosen that the higher approximations shall be no more singular than the first. This remark will become clear in the example discussed below. This method with various applications is presented by Tsien [15] who called it the Poincaré–Lighthill–Kuo or, in short, the PLK method because of the contribution of Poincaré [16] and Lighthill [14] to this method and its extensive application by Kuo [17]. The method has also been referred to as the method of strained coordinates (see Van Dyke [18], Chapter 6), where many examples of the applications of this method in aerodynamics are also presented. Lighthill [19] has applied the method to conical shock waves in steady supersonic flows and Legras [20, 21] to supersonic air foils.

The PLK method is also applicable to cases where the nonuniformity of the solution arises in higher than the zeroth-order regular perturbation solution. While studying the nonlinear problem of temperature distribution in a melting slab with the melting face subject to a constant heat flux, the farther end of the slab being insulated, Goodman and Shea [22], by an integral method, derived the system of nonlinear coupled differential equations

$$\frac{ds}{d\theta} + \mu \frac{dw}{d\theta} + \mu \frac{dv}{d\theta} = 2\mu\alpha_1, \tag{3}$$

$$\frac{s^2}{3} \frac{dw}{d\theta} = \alpha_1 s^2 - vw, \tag{4}$$

$$(1 - s)^2 \frac{dv}{d\theta} = -3v, \tag{5}$$

to be solved with the initial conditions

$$s(0) = 0, \qquad w(0) = 0, \qquad v(0) = -2\alpha_1/3, \tag{6}$$

where s, w, v represent the nondimensional parameters related to the position of the melting interface and the temperature in the melted and unmelted parts of the slab, respectively, and α_1 is a parameter with natural restriction $0 < \alpha_1 < 1$. If a regular perturbation scheme is adopted for the solution of the above system of equations, namely, if we assume an expansion of the following type

$$f(\alpha_1, \mu, \nu, \theta) = f_0(\mu, \nu, \theta) + \alpha_1 f_1(\mu, \nu, \theta) + \alpha_1^2 f_2(\mu, \nu, \theta) \cdots \tag{7}$$

for each of the dependent variables s, w, v, it is found that the second-order solution $v_2(\theta)$ contains terms of the type $\theta \exp(-3\theta)$ and $\theta^2 \exp(-3\theta)$. Thus the value of v obtained by this method for large θ achieves positive values and for moderately large θ these terms create humps in the solution. Since it follows from Eq. (5) and the last of the initial conditions in Eq. (6) that v has to be negative, the entire solution obtained by the regular perturbation scheme is invalidated.

To obtain a uniformly valid solution, we make recourse to the PLK method. Thus we define a new independent variable ζ and expand s, v, w and the independent variable θ in terms of the new independent variable ζ in powers of the small parameter α. Thus we assume

$$\theta(\alpha_1, \mu, \nu, \zeta) = \zeta + \alpha_1 \theta_1(\mu, \nu, \zeta) + \alpha_1^2 \theta_2(\mu, \nu, \zeta) + \cdots \tag{8}$$

and expansions similar to Eq. (7) with θ replaced by the new independent variable ζ for s, w, v. Thus the problem posed by Eqs. (3)–(6) can be restated as

$$\frac{ds}{d\zeta} + \mu \frac{dw}{d\zeta} + \mu \frac{dv}{d\zeta} = 2\mu\alpha_1 \frac{d\theta}{d\zeta}, \tag{9}$$

$$\frac{s^2}{3} \frac{dw}{d\zeta} = (\alpha_1 s^2 - vw) \frac{d\theta}{d\zeta}, \tag{10}$$

$$\frac{(1-s)^2}{3} \frac{dv}{d\zeta} = -v \frac{d\theta}{d\zeta} \tag{11}$$

with initial conditions

$$s(\zeta = 0) = 0, \qquad w(\zeta = 0) = 0, \qquad v(\zeta = 0) = -2\alpha_1/3, \tag{12}$$

provided $\theta_i(\zeta)$ in Eq. (8) are so chosen that $\theta_i(\zeta = 0) = 0$, $i \geqslant 1$.

Introducing the expansion in Eq. (8) and similar expansions for s, w, v in the system of equations (9)–(12), the zeroth-order solution is given as

$$s_0(\zeta) = w_0(\zeta) = v_0(\zeta) = 0, \tag{13}$$

and the first-order solution as

$$s_1(\zeta) = 2\mu\zeta + (2\mu/3)[\exp(-3\zeta) - 1], \tag{14}$$

$$w_1(\zeta) = 0, \tag{15}$$

$$v_1(\zeta) = -\tfrac{2}{3} \exp(-3\zeta). \tag{16}$$

The equations for the second-order approximation become

$$\frac{ds_2}{d\zeta} + \mu\frac{dw_2}{d\zeta} + \mu\frac{dv_2}{d\zeta} = 2\mu\frac{d\theta_1}{d\zeta}, \tag{17}$$

$$w_2 = 0, \tag{18}$$

$$\begin{aligned}\frac{dv_2}{d\zeta} + 3v_2 &= 2s_1\frac{dv_1}{d\zeta} - 3v_1\frac{d\theta_1}{d\zeta} \\ &= 2\exp(-3\zeta)\left\{4\mu[\zeta - \tfrac{1}{3} + \tfrac{1}{3}\exp(-3\zeta)] + \frac{d\theta_1}{d\zeta}\right\}.\end{aligned} \tag{19}$$

It is clear from the above that if the right-hand side of Eq. (19) is free from terms of the type $\exp(-3\zeta)$ and $\zeta\exp(-3\zeta)$; v_2 would no longer contain the undesired terms $\zeta\exp(-3\zeta)$ and $\zeta^2\exp(-3\zeta)$. Thus θ_1 should be chosen such that

$$\frac{d\theta_1}{d\zeta} + 4\mu(\zeta - \tfrac{1}{3}) = 0,$$

or

$$\theta_1 = -2\mu(\zeta^2 - \tfrac{2}{3}\zeta). \tag{20}$$

With this choice of θ_1 we get

$$s_2 = -4\mu^2(\zeta^2 - \tfrac{2}{3}\zeta) - \tfrac{8}{9}\mu^2[\exp(-3\zeta) - \exp(-6\zeta)], \tag{21}$$

$$v_2 = \tfrac{8}{9}\mu[\exp(-3\zeta) - \exp(-6\zeta)]. \tag{22}$$

The corresponding third-order approximation is given by

$$v_3 = \tfrac{4}{27}\mu^2[-9\exp(-3\zeta) + 20\exp(-6\zeta) - 11\exp(-9\zeta)], \tag{23}$$

$$s_3 = 2\mu\theta_2 - (\mu/v)s_1{}^2 - \mu v_3, \tag{24}$$

$$w_3 = s_1{}^2/v, \tag{25}$$

where the choice of θ_2 as dictated by this method is

$$\theta_2 = 4\mu^2\{(\zeta - \tfrac{1}{3})^3 + \tfrac{2}{9}(\zeta + \tfrac{1}{3})[\exp(-3\zeta) - 1] + \tfrac{1}{27}\}. \tag{26}$$

In this manner we have been able to obtain a uniformly valid solution for all ζ.

Ahuja and Kumar [23] have applied the same technique for rendering uniformly valid the solution of the problem of the temperature distribution in a melting cylindrical tube, while the method has been used by Morris [24] to obtain a uniformly valid solution of the laminar convective flow in a heated vertical tube rotating about a parallel axis. Olstad [25] considered the problem of radiating flow near a stagnation point as a perturbation of the case without radiation. It was found that near the wall the regular perturbation procedure failed and therefore the PLK method was used for obtaining a uniformly valid solution.

B. METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

Where the highest derivative in a differential equation is multiplied by the small parameter the PLK method fails. The basic difficulty in such problems arises from the fact that when the order of the equation is reduced certain boundary conditions cannot be satisfied. For such problems the works of Lagerstrom and Cole [26], Lagerstrom [27], and Kaplun [28, 29] developed the method of matched asymptotic expansions.

Let $v(x, \epsilon)$ be the solution of the singular perturbation problem. The usual asymptotic expansion in powers of ϵ, $\epsilon \to 0$ is called the outer expansion for $x > 0$ fixed. This expansion is valid in the interval $\gamma \leqslant x \leqslant 1$ with γ independent of ϵ. The expansion may also hold for $\gamma \leqslant x \leqslant 1$ even if γ depends on ϵ and approaches zero as $\epsilon \to 0$, provided $\gamma/\epsilon \to \infty$. Let the outer expansion be denoted by v°. To obtain the inner expansion a stretching transformation $x = Z\epsilon$ is introduced [30]. The asymptotic expansion $v(Z\epsilon, \epsilon)$, for $\varepsilon \to 0$ while $Z \geqslant 0$ is fixed, is called the inner expansion denoted by v^{i}. This expansion is valid for $0 \leqslant (Z = x/\epsilon) \leqslant \delta$. The inner and outer expansions thus have a common region of validity and in this region we can write the inner expansion of the outer expansion $(v^{\mathrm{o}})^{\mathrm{i}}$ and the outer expansion of the inner expansion $(v^{\mathrm{i}})^{\mathrm{o}}$. The asymptotic matching principle (Van Dyke [18], p. 64) states that:

$$\frac{\text{The } m\text{-term inner expansion}}{\text{(of the } n\text{-term outer expansion)}} = \frac{\text{the } n\text{-term outer expansion}}{\text{(of the } m\text{-term inner expansion)}}, \tag{27}$$

where m and n are any two integers. In practice m is usually chosen either equal to n or $n + 1$. The unknown constants in v^{o} and v^{i} are determined by matching the two in the common region of validity with the help of the above stated matching principle. Sometimes a composite expansion v^{c} is formed to obtain a solution uniformly valid throughout the interval $0 \leqslant x \leqslant 1$. v^{c} can be formed either according to the additive law

$$v^{\mathrm{c}} = v^{\mathrm{o}} + v^{\mathrm{i}} - (v^{\mathrm{o}})^{\mathrm{i}} \tag{28a}$$

or the multiplicative law as detailed in [18, p. 94):

$$v^{\mathrm{c}} = v^{\mathrm{o}}v^{\mathrm{i}}/(v^{\mathrm{o}})^{\mathrm{i}}. \tag{28b}$$

A very instructive application of the method of matched asymptotic expansions has been made by Inger [31] in the analysis of near-equilibrium dissociating boundary layers. We describe briefly part of this work to illustrate the method.

1. *An Example*

Consider the near equilibrium dissociating boundary layer flow of a diatomic gas along an impervious, axisymmetric or two-dimensional body which is either adiabatic or has a uniform surface temperature. Introducing the variables

$$\eta = \rho_e u_e r_B^j (2\xi)^{-\frac{1}{2}} \int_0^y (\rho/\rho_e)\, dy, \tag{29}$$

$$\xi = C \int_0^x \rho_e \mu_e u_e r_B^{2j}\, dx, \tag{30}$$

$$u = u_e\, df/d\eta = u_e f', \tag{31}$$

and assuming the Prandtl number and the Lewis number to be unity and $\rho\mu = \text{const.}$, we can write down the equations of momentum, atom concentration, and energy in the form

$$ff'' + f''' = 0, \tag{32}$$

$$f\frac{\partial \alpha}{\partial \eta} + \frac{\partial^2 \alpha}{\partial \eta^2} - 2\xi f' \frac{\partial \alpha}{\partial \xi} = \bar{\Gamma}\xi^R(\alpha - C_1 - C_2 t), \tag{33}$$

$$f\frac{\partial H}{\partial \eta} + \frac{\partial^2 H}{\partial \eta^2} - 2\xi f' \frac{\partial H}{\partial \xi} = 0. \tag{34}$$

The total enthalpy H is related to the static temperature t and atom mass fraction α by

$$H = c_p t + \alpha h_D + \tfrac{1}{2}u_e^2 (f')^2. \tag{35}$$

It may be noted that $\bar{\Gamma} \to 0$ for chemically frozen flow and $\bar{\Gamma} \to \infty$ for complete equilibrium. The boundary conditions are

$$\begin{gathered} f'(\infty) = 1, \qquad \alpha(\xi,\infty) = \alpha_e = C_1 + C_2 t_e, \qquad t(\xi,\infty) = t_e \\ H(\xi,\infty) = H_e = c_p t_e + \alpha_e h_D + \tfrac{1}{2}u_e^2. \end{gathered} \tag{36}$$

At the surface

$$f(0) = f'(0) = 0, \tag{37}$$

$$t(\xi,0) = t_w = \text{const.} \quad \text{or} \quad \partial H(\xi,0)/\partial \eta = 0, \tag{38}$$

$$H(\xi,0) = c_p t_w + h_D \alpha(\xi,0). \tag{39}$$

For a perfectly catalytic wall we also have

$$\alpha(\xi,0) = \alpha_{EQ,w} = C_1 + C_2 t_w. \tag{40}$$

Thus in Eq. (33) for $\bar{\Gamma} \to \infty(1/\bar{\Gamma} \to 0)$, all derivatives including the highest order vanish and the problem is therefore a singular perturbation problem. We may write Eqs. (33) and (34) in terms of the new dependent variables $\bar{\alpha}$, G and the new parameter Γ as

$$f\frac{\partial\bar{\alpha}}{\partial\eta} + \frac{\partial^2\bar{\alpha}}{\partial\eta^2} + 2\xi f'\frac{\partial\bar{\alpha}}{\partial\xi} = \Gamma\xi^R\left(\bar{\alpha} - \frac{DG}{1+D}\right) - (f\alpha'_{EQ} + \alpha''_{EQ}), \tag{41}$$

$$f\frac{\partial G}{\partial\eta} + \frac{\partial^2 G}{\partial\eta^2} - 2\xi f'\frac{\partial G}{\partial\xi} = 0, \tag{42}$$

subject to the following boundary conditions for the catalytic wall:

$$\bar{\alpha}(\xi,\infty) = 0 = G(\xi,0), \tag{43}$$

$$\bar{\alpha}(\xi,0) = 0 = G(\xi,0). \tag{44}$$

Noting that the temperature profile can be written as

$$t(\xi,\eta) = t_{EQ}(\eta) + c_p^{-1}Gh_D - h_D c_p^{-1}\bar{\alpha}, \tag{45}$$

it follows from Eq. (42) that

$$G(\xi,\eta) = 0. \tag{46}$$

a. Outer Expansion. Consider Eq. (41) for near-equilibrium flow where Γ is very large. The outer expansion can therefore be assumed for $\bar{\alpha} - D(1 + D)^{-1}G$ in the form

$$\bar{\alpha} - D(1 + D)^{-1}G = \sum_{N=1}^{\infty}\bar{\alpha}_N^o(\eta)(\Gamma\xi^R)^{-N}. \tag{47}$$

Substituting from Eq. (47) in Eq. (41), using Eq. (42) and collecting terms in like powers of Γ, we determine the following equations governing the perturbation functions:

$$\bar{\alpha}_1^o(\eta) = f\alpha'_{EQ} + \alpha''_{EQ} = \alpha''_{EQ}(0)[f''(\eta)/A]^2, \tag{48}$$

$$\bar{\alpha}_2^o(\eta) = f(\bar{\alpha}_1^o)' + (\bar{\alpha}_1^o)'' + 2Rf'\bar{\alpha}_1^o \tag{49}$$

$$\bar{\alpha}_N^o(\eta) = f(\bar{\alpha}_{N-1}^o)' + (\bar{\alpha}_{N-1}^o)'' + 2R(N-1)f'\bar{\alpha}_{N-1}^o, \tag{50}$$

where Eq. (45) has been used to simplify the right-hand side of Eq. (48). Although the expansion in Eq. (47) satisfies the outer boundary condition Eq. (43) it cannot satisfy the inner boundary condition Eq. (44). We therefore try to obtain the solution near the wall in a contracted variable Q.

b. Inner Expansion. From Eq. (41) it follows that the coëfficient of the highest order derivative on the left-hand side would not vanish for $\Gamma \to \infty$ if we use a new independent variable $Q = \Gamma^{1/2}\eta$. To obtain the inner solution in terms of the new independent variable we first rewrite Eq. (41) in terms of this variable. Thus we get

$$\frac{\partial^2\bar{\alpha}}{\partial Q^2} + \frac{AQ^2}{2\Gamma^{3/2}}\left(1 - \frac{2BQ^3}{A\Gamma^{3/2}}\right)\frac{\partial\bar{\alpha}}{\partial Q} - \frac{2\xi AQ}{\Gamma^{3/2}}\left(1 - \frac{5BQ^3}{A\Gamma^{3/2}}\right)\frac{\partial\bar{\alpha}}{\partial\xi}$$

$$= \xi^R\left[\bar{\alpha} - \frac{DG(\xi,0)}{1+D} - \frac{DQ}{1+D}\Gamma^{-1/2}\frac{\partial G(\xi,0)}{\partial\eta}\right] - \alpha''_{EQ}(0)/\Gamma \qquad (51)$$

subject to the wall boundary condition Eq. (44). Equation (51) is to be solved subject to Eq. (44). Let us assume that

$$\bar{\alpha} = \sum_{N=1}^{\infty} \alpha_N^i(\xi,Q)\Gamma^{-N/2} . \qquad (52)$$

Substitution of the series Eq. (52) in Eq. (51) yields a sequence of linear second-order differential equations governing the inner perturbation functions. Solving these equations we get

$$\bar{\alpha}_1^i(\xi,Q) = E_1 \sinh(\xi^{R/2} Q), \qquad (53)$$

$$\bar{\alpha}_2^i(\xi,Q) = E_2 \sinh(\xi^{R/2} Q) - \xi^{-R}\alpha''_{EQ}(0)[1 - \exp(-\xi^{R/2} Q)], \qquad (54)$$

$$\bar{\alpha}_3^i(\xi,Q) = E_3 \sinh(\xi^{R/2} Q), \qquad (55)$$

where $E_1, E_2, \ldots, E_n$ are arbitrary constants to be determined by matching the outer and the inner solutions.

c. Matching. From Eqs. (46)–(50) we have the outer solution for the atom concentration

$$\bar{\alpha}^o(\eta,\xi) = [\alpha''_{EQ}(0)/(\Gamma\xi^R)][f''(0)/A]^2 + O(\Gamma^{-2}\xi^{-2R}). \qquad (56)$$

Rewriting in terms of the inner variable Q, expanding for large Γ, and using the fact that $f'''(0) = 0$, Eq. (56) becomes

$$\bar{\alpha}^o(Q,\xi) = \alpha''_{EQ}(0)(\Gamma\xi^R)^{-1} + O(\Gamma^{-2}\xi^{-2R}) \qquad (57)$$

From Eqs. (52)–(55) the inner solution is

$$\bar{\alpha}^i(Q,\xi) = E_1\Gamma^{-1/2}\sinh(\xi^{R/2}Q) + E_2\Gamma^{-1}\sinh(\xi^{R/2}Q)$$
$$+ \alpha''_{EQ}(0)[1 - \exp(-\xi^{R/2} Q)]\Gamma^{-1}\xi^{-R}$$
$$+ E_3\Gamma^{-3/2}\sinh(\xi^{R/2}Q) + O(\Gamma^{-2}), \qquad (58)$$