IAN ANDERSON

# A first course in combinatorial mathematics

SECOND EDITION

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*University of Glasgow*

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# Preface to the second edition

THE publication of a second edition provides the opportunity of making a number of changes to the original text. Many of these changes are made in response to comments from university teachers who have used the first edition as a recommended textbook. More exercises have been provided, particularly in Chapters 2, 4, and 6; some of the material on recurrence relations has been expanded; an application of the marriage theorem to score sequences of tournaments has been included; and a gap in the presentation of the optimal assignment problem in Section 3.3 has been filled. The greatest changes occur in Chapter 7: Steiner triple systems are constructed and $S(5, 8, 24)$ is obtained via the Golay code of length 24 rather than by the (incomplete) construction of the first edition. The last chapter therefore combines together three combinatorial structures; the Leech lattice, the Golay codes, and Steiner systems.

The spirit and the aim of the book remain as before, to present a compact introduction to a spread of both enumerative and constructive topics which will give the reader a flavour of the distinctive characteristics of this attractive and increasingly important branch of mathematics.

*Glasgow*
*July 1988*

I.A.

# Preface to the first edition

IN 1857, the Rev. Thomas Kirkman presented to the Historic Society of Lancashire and Cheshire a paper which had nothing to do with either history or theology. The paper was concerned with a problem in what is now known as *combinatorial mathematics*, and Kirkman began by explaining the attraction of such problems. 'The elements to be combined in these questions', he remarked, 'have no property except that of diversity. They have no arithmetic value or capacity, except that they can be counted. No operation of addition, subtraction, multiplication, or division can be performed upon them. They can merely be combined.'

Combinatorial mathematics, then, is first of all concerned with counting the number of ways of arranging given objects in a prescribed way. The aim of the first part of this book is to introduce the reader to a working knowledge of the basic ideas and techniques of the subject. Chapters 2, 3, 4, and 5 contain essential technical know-how, Chapter 3 also being concerned with various aspects of *assignment* problems, beginning with the famous result of Philip Hall, and leading on to various applications. In an attempt to give a balanced view of the subject, the remaining chapters deal with configurations rather than techniques. The emphasis here is not so much on the question 'how many . . .?' but on the structure and properties of systems satisfying certain prescribed conditions. A study of block designs suggests applications to error-correcting codes, and in the final chapter a study of the Steiner system $S(5, 8, 24)$ leads on to the construction of the well-known Leech lattice in 24 dimensions. Thus some idea is given of how the properties of seemingly useless systems can be put to use in interesting applications.

This little book is not intended as an encyclopaedia for research workers: such people are well served already. It is intended as a textbook for anyone to work from who wishes to become acquainted with the flavour of the subject and the basic tools of the trade. The subject has come a long way since Kirkman's time, but it still remains an easily accessible area of mathematics, one which is becoming more and more widely used in other disciplines. The days are past when the calculus was thought to be the queen of applicable mathematics. But, despite its applications, the subject of this book is genuine mathematics in all its purity, and as such is worthy of study just for its own attractiveness and charm.

The reader who is new to the subject should not be put off by the lengthy list of references to mathematical papers at the end of the book. An understanding of the text in no way requires a study of these, but it is hoped that the reader will, as well as becoming aware of how contemporary the subject is, follow up one or two of these references and see for himself the type of work which is going on today.

A number of people have helped in the preparation of this book. My thanks go to Dr H. G. Martin and Dr J. R. Gillett who encouraged me to start writing, and to the Clarendon Press for their encouragement in the later stages. Finally I extend my gratitude to all those who first interested me in combinatorial ideas, and to all the mathematicians whose work in the subject is now the standard repertoire.

*Glasgow, January 1973* I.A.

# Contents

# 1 Introduction to basic ideas

TO embark upon a mass of unmotivated theory is perhaps the simplest and quickest way of losing the reader. This study of combinatorial ideas will therefore begin with a specific problem, and discover several methods for its solution, introducing the reader incidentally to some of the basic ideas on which the remainder of the book will build. Each of these approaches will be examined later in greater depth to see how they can be applied in other situations or modified to yield new techniques.

The specific problem to be considered is the following.

## Problem

Suppose that each of $k$ indistinguishable golf balls has to be coloured with any one of $n$ given colours. How many different colourings are possible?

If $x_1$ denotes the number of balls coloured with the first colour, $x_2$ the number coloured with the second colour, and so on, the required number is the number of solutions of the equation

$$x_1 + x_2 + \cdots + x_n = k,$$

in non-negative integers $x_1, \ldots, x_n$. As this number will depend upon both $n$ and $k$, denote it by $f(n, k)$.

*Special cases*

(a) If there is only one colour available (i.e. $n = 1$) the $k$ balls can be coloured in only one way. Thus

$$f(1, k) = 1 \quad \text{for all } k \geqslant 1. \tag{1.1}$$

(b) If there are $n$ colours but only one ball, there are $n$ possible colourings. Thus

$$f(n, 1) = n \quad \text{for all } n \geqslant 1. \tag{1.2}$$

(c) If $n$ and $k$ are both very small, then $f(n, k)$ can be found without much difficulty. For example, suppose $n = k = 2$. If the colours are black and white, the balls can be coloured both black, both white, or one of each. Thus $f(2, 2) = 3$.

The problem is to find a general formula for $f(n, k)$. Three possible approaches will be mentioned.

*First approach*

It has already been pointed out that if $n$ and $k$ are both small then $f(n, k)$ can be found quite easily. This suggests that one method of attack would be to try to express $f(n, k)$ in terms of similar expressions with smaller values of $n$ and $k$. For example, if it were known that

$$f(3, 2) = f(2, 2) + f(3, 1)$$

it could be deduced that

$$f(3, 2) = 3 + 3 = 6,$$

using results already known. So consider $f(n, k)$ and concentrate on the $n$th colour. When the $k$ balls are coloured, this $n$th colour may or may not be used. If it is not used, there are in effect only $(n - 1)$ colours and so the colourings can be performed in $f(n - 1, k)$ ways. But if the $n$th colour is used, one ball can be coloured by it and then removed to leave $(k - 1)$ balls which can be coloured in $f(n, k - 1)$ ways. Thus

$$f(n, k) = f(n - 1, k) + f(n, k - 1). \tag{1.3}$$

A relation such as (1.3) is called a *recurrence relation* and our ability to solve the original colouring problem now depends on our ability to solve recurrence relations. By solving (1.3) is meant finding the *unique* function $f$ which satisfies (1.3) and the *boundary conditions* (1.1) and (1.2). In general, a recurrence relation has more than one possible solution, but the boundary conditions specify which of these is the required solution.

Recurrence relations are important in combinatorial mathematics and are widespread in their appearances. Perhaps the most famous of all occurs in the definition of the *Fibonacci sequence* $\{a_n\}$,

$$1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

given by the boundary conditions

$$a_1 = 1, \quad a_2 = 2,$$

and the recurrence relation

$$a_n = a_{n-1} + a_{n-2} \quad (n \geqslant 3).$$

From the third term onwards each is the sum of the two preceding terms. Solving this recurrence relation means obtaining a formula for the $n$th term $a_n$ as a function of $n$. This will be done later.

**Example 1.1.** Solve the recurrence relation $a_n = na_{n-1}$, subject to the condition $a_1 = 1$.

*Solution.* The first few $a_n$ can easily be found by working up from $a_1$. Thus $a_2 = 2a_1 = 2$, $a_3 = 3a_2 = 6$ and so on. But to find a formula for $a_n$ it is best to start at the top and work down. Thus $a_n = na_{n-1}$ and similarly $a_{n-1} = (n-1)a_{n-2}$, so that

$$a_n = n(n-1)a_{n-2} = n(n-1)(n-2)a_{n-3}$$

and finally

$$\begin{aligned} a_n &= n(n-1)(n-2)\ldots 2a_1 \\ &= n(n-1)(n-2)\ldots 2.1 \\ &= n!, \end{aligned}$$

where $n!$ ('$n$ factorial') is the product of the first $n$ positive integers. Thus, for example,

$$a_3 = 3! = 3.2.1 = 6, \quad a_5 = 5! = 120.$$

## Exercises 1.1

1. Show that $f(4, 2) = 10$ and $f(5, 3) = 35$.
2. Solve the recurrence relation $a_n = n^2 a_{n-1}$ given that $a_1 = 1$.
3. If $a_n = \dfrac{n-1}{n} a_{n-1}$ and $a_1 = 2$, find $a_n$.
4. If $a_n = a_{n-1} + n$, find $a_n$ if (a) $a_1 = 1$, (b) $a_1 = 0$.
5. Suppose a row of $n$ cages is given, and it is required to place $k$ indistinguishable lions in them so that no cage contains more than one lion, and no two consecutive cages both contain a lion. Let $g(n, k)$ denote the number of ways in which this can be done. Prove that

   (a) $g(2k-1, k) = 1$,
   (b) $g(n, k) = 0$ if $n < 2k - 1$,
   (c) $g(n, 1) = n$,
   (d) $g(n, k) = g(n-2, k-1) + g(n-1, k)$ if $k \geqslant 2$.

   (Hint: consider whether or not the last cage contains a lion.) Deduce that

   (e) $g(6, 3) = 4$,
   (f) $g(2k, k) = g(2k-2, k-1) + 1$,
   (g) $g(2k, k) = k + 1$.

6. Suppose it is known that $t(n, n-1) = 1$ and that

$$(n-k-1)t(n, k) = k(n-1)t(n, k+1)$$

for each $k < n-1$. Deduce that

$$t(n, k) = \frac{(n-1)^{n-k-1}(n-2)!}{(k-1)!(n-k-1)!}.$$

*Second approach*

Consider the effect of multiplying the following expressions together,

$$(1+x+x^2+x^3+\cdots)(1+x+x^2+\cdots)\ldots(1+x+x^2+\cdots), \tag{1.4}$$

where there are $n$ brackets. Different powers of $x$ are obtained, each power occurring a number of times. For example, $x^2$ can be obtained in $n$ ways by multiplying an $x^2$ term from one bracket by the terms 1 in each of the remaining brackets, and in yet more ways by multiplying two $x$ terms from two of the brackets by the terms 1 in each of the remaining brackets.

More generally, how often will the term $x^k$ appear, i.e. what will the coefficient of $x^k$ be in the resulting expression? If the convention $x^0 = 1$ is followed, $x^k$ can be obtained by choosing a term $x^{t_1}$ in the first bracket, $x^{t_2}$ in the second, and so on, with the condition that

$$t_1 + t_2 + \cdots + t_n = k.$$

Thus the coefficient of $x^k$ will be precisely the number $f(n, k)$. It follows that

$$\begin{aligned} f(n, k) &= \text{coefficient of } x^k \text{ in } (1.4) \\ &= \text{coefficient of } x^k \text{ in } (1+x+x^2+\cdots)^n. \end{aligned}$$

But

$$1+x+x^2+\cdots = \frac{1}{1-x} = (1-x)^{-1},$$

so that

$$f(n, k) = \text{coefficient of } x^k \text{ in } (1-x)^{-n}. \tag{1.5}$$

This approach therefore reduces the problem to an application of the *binomial theorem*. This theorem will be studied in the next chapter, when it will be seen that

$$(1-x)^{-n} = 1 + \frac{n}{1}x + \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n+2)}{1.2.3}x^3 + \cdots$$

It will then follow that

$$f(n, k) = \frac{n(n+1)(n+2)\ldots(n+k-1)}{1.2.3.\ldots.k}$$

$$= \frac{n(n+1)\ldots(n+k-1)}{k!}$$

$$= \frac{(n+k-1)!}{(n-1)!\,k!}. \qquad (1.6)$$

For example,

$$f(9, 4) = \frac{12!}{8!\,4!} = \frac{12.11.10.9}{4.3.2.1} = 495.$$

The technique of picking out the coefficient of a particular power of $x$ leads on to the general technique of *generating functions*. If a fixed value of $n$ is chosen in the above example, then, by (1.5),

$$(1-x)^{-n} = f(n, 0) + f(n, 1)x + f(n, 2)x^2 \ldots$$

since $f(n, k)$ is precisely the coefficient of $x^k$ in the expansion of $(1-x)^{-n}$. $(1-x)^{-n}$ is called the generating function for the numbers $f(n, k)$ since its expansion generates these numbers as coefficients. Again, generating functions will be seen to be a useful tool.

### *Third approach*

The problem is to colour $k$ balls using $n$ colours. One way of looking at this is to think of the colouring process as a splitting up of the $k$ balls into $n$ smaller collections some of which might be empty, where in the first collection $x_1$ balls are chosen and coloured with the first colour, and so on. For example, suppose that $n = 4$ and $k = 5$. One possible colouring would be to colour 2 balls with the first colour and one with each of the remaining colours. This corresponds to $x_1 = 2, x_2 = x_3 = x_4 = 1$, and can be represented geometrically as follows. Put two crosses at consecutive marks on a straight line. These crosses represent the two balls coloured with the first colour. At the next mark on the line place a O to signify a change of colour. Then put one cross to represent one ball of the second colour, then another O, and so on, to obtain

$$\times \times O \times O \times O \times$$

which has 5 Xs (the 5 balls) and 3 Os (three changes of colour). Similarly, corresponding to $x_1 = 2, x_2 = 0, x_3 = 2, x_4 = 1$,

$$\times \times O O \times \times O \times$$

would be obtained. In the general case, with $n$ colours and $k$ balls, $f(n, k)$ will be the number of such sequences of Os and Xs containing exactly $k$ Xs and $(n-1)$Os. Each such sequence contains $(n + k - 1)$ symbols altogether, and is uniquely determined once the Os have been positioned. Thus

$f(n, k)$ = number of ways of choosing $(n-1)$ places out of $(n + k - 1)$ places.

This third approach therefore leads to the following general question. Given $m$ objects, how many ways are there of choosing exactly $r$ of them? This is perhaps the most basic question in the whole subject, and so the next chapter will take a careful look at the problem of selections.

## Exercises 1.2

1. Use (1.6) to evaluate $f(10, 3), f(10, 4), f(10, 5)$.

2. Verify that $f$, as given by (1.6), satisfies (1.1), (1.2), and (1.3).

3. By listing all the possibilities, show that there are 10 ways of choosing 3 girls from 5 given girls.

4. In a football league of $n$ teams, each team plays each other twice. The number of games played is therefore $2c$ where $c$ is the number of ways of choosing 2 objects from $n$ given objects. Prove that
$$c = (n-1) + (n-2) + \cdots + 3 + 2 + 1 = \tfrac{1}{2}n(n-1),$$
and deduce the number of games played in a league of 22 teams.

5. Let $h(n, k)$ denote the number of ways in which $k$ indistinguishable golf balls can be coloured with $n$ colours so that there is at least one ball of each colour. Prove that

   (a) $h(n, k) = 0$ if $n > k$,
   (b) $h(n, k)$ is the coefficient of $x^k$ in $(x + x^2 + x^3 + \cdots)^n$,

(c) $h(n, k)$ is the coefficient of $x^{k-n}$ in $(1 - x)^{-n}$,
(d) $h(n, k) = f(n, k - n)$,
(e) $h(n, k) = \dfrac{(k-1)!}{(n-1)!\,(k-n)!}$ if $n \leqslant k$ (use (1.6) here).

Hence find $h(5, 10)$.

6. Expand $(1 - x)^{-3}$ up to the term in $x^5$.

7. Express, in terms of the function $f$ of page 1:

(a) the number of binary sequences (i.e. sequences of 0s and 1s) of length 10 containing exactly 5 0s;
(b) the number of solutions in (i) non-negative, (ii) positive integers of the equation

$$x + y + z = 24;$$

(c) the number of routes in the $xy$-plane from $(0, 0)$ to $(m, n)$ consisting of a succession of steps each of which involves increasing the $x$ or the $y$ coordinate by 1;
(d) the number of different number patterns obtainable on throwing three indistinguishable dice.