MECHANICS OF SOLIDS

VOLUME II

Linear Theories of Elasticity and Thermoelasticity
Linear and Nonlinear Theories of Rods, Plates, and Shells

Editor
C. Truesdell

China Academic Publishers, Beijing

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Editor

C. Truesdell

Contributions by

S. S. Antman D. E. Carlson G. Fichera
M. E. Gurtin P. M. Naghdi

With 25 Figures

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Volume Editor
Professor Dr. C. TRUESDELL
The Johns Hopkins University, Baltimore, MD 21218, USA

Chief Editor of Encyclopedia of Physics
Professor Dr. S. FLÜGGE
Fakultät für Physik, Universität Freiburg
D-7800 Freiburg, Fed. Rep. of Germany

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Preface.

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The mechanical response of solids was first reduced to an organized science of fairly general scope in the nineteenth century. The theory of small elastic deformations is in the main the creation of CAUCHY, who, correcting and simplifying the work of NAVIER and POISSON, through an astounding application of conjoined scholarship, originality, and labor greatly extended in breadth the shallowest aspects of the treatments of particular kinds of bodies by GALILEO, LEIBNIZ, JAMES BERNOULLI, PARENT, DANIEL BERNOULLI, EULER, and COULOMB. Linear elasticity became a branch of mathematics, cultivated wherever there were mathematicians. The magisterial treatise of LOVE in its second edition, 1906 - clear, compact, exhaustive, and learned - stands as the summary of the classical theory. It is one of the great "gaslight works" that in BOCHNER's words[1] "either do not have any adequate successor[s] … or, at least, refuse to be superseded …; and so they have to be reprinted, in ever increasing numbers, for active research and reference", as long as State and Society shall permit men to learn mathematics by, for, and of men's minds.

Abundant experimentation on solids was done during the same century. Usually the materials arising in nature, with which experiment most justly concerns itself, do not stoop easily to the limitations classical elasticity posits. It is no wonder that the investigations LOVE's treatise collects, condenses, and reduces to symmetry and system were in the main ill at ease with experiment and unconcerned with practical applications. In LOVE's words, they belong to "an abstract conceptual scheme of Rational Mechanics". He concluded thus his famous Historical Introduction:

The history of the mathematical theory of Elasticity shows clearly that the development of the theory has not been guided exclusively by considerations of its utility for technical Mechanics. Most of the men by whose researches it has been founded and shaped have been more interested in Natural Philosophy than in material progress, in trying to understand the world than in trying to make it more comfortable. From this attitude of mind it may possibly have resulted that the theory has contributed less to the material advance of mankind than it might otherwise have done. Be this as it may, the intellectual gain which has accrued from the work of these men must be estimated very highly. The discussions that have taken place concerning the number and meaning of the elastic constants have thrown light on most recondite questions concerning the nature of molecules and the mode of their interaction. The efforts that have been made to explain optical phenomena by means of the hypothesis of a medium having the same physical character as an elastic solid body led in the first instance, to

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[1]SALOMON BOCHNER: "Einstein between Centuries", Rice Univ Stud. 65 (3), 54 (1979).

the understanding of a concrete example of a medium which can transmit transverse vibrations, and, at a later stage, to the definite conclusion that the luminiferous medium has not the physical character assumed in the hypothesis. They have thus issued in an essential widening of our ideas concerning the nature of the aether and the nature of luminous vibrations. The methods that have been devised for solving the equations of equilibrium of an isotropic solid body form part of an analytical theory which is of great importance in pure mathematics. The application of these methods to the problem of the internal constitution of the Earth has led to results which must influence profoundly the course of speculative thought both in Geology and in cosmical Physics. Even in the more technical problems, such as the transmission of force and the resistance of bars and plates, attention has been directed, for the most part, rather to theoretical than to practical aspects of the questions. To get insight into what goes on in impact, to bring the theory of the behaviour of thin bars and plates into accord with the general equations – these and such-like aims have been more attractive to most of the men to whom we owe the theory than endeavours to devise means for effecting economies in engineering constructions or to ascertain the conditions in which structures become unsafe. The fact that much material progress is the indirect outcome of work done in this spirit is not without significance. The equally significant fact that most great advances in Natural Philosophy have been made by men who had a first-hand acquaintance with practical needs and experimental methods has often been emphasized; and, although the names of Green, Poisson, Cauchy show that the rule is not without important exceptions, yet it is exemplified well in the history of our science.

Love's treatise mentions experiment rarely and scantly. Its one passage concerning experiment in general, § 63, in effect warns its reader to have a care of experimental data because of their indirectness.

In an irony of history the ever-increasing use of mathematical notation in physical science, to the point that now often works on experiment are dominated by their authors' seemingly compulsive recourse to mathematical formulae interconnected by copied or adapted bits of old mathematical manipulation, Love's treatise is sometimes in reproaches upon modern "pure" or "abstract" researchers held up as a model of practical, applied theory.

Experiment on the mechanical properties of solids became in the later nineteenth century a science nearly divorced from theory. Nevertheless, no great treatise on experiment fit to be set beside Love's on theory ever appeared. Even such books of experiment as were published seem to have in the main taken positions either dominated by theory, usually crude, verbose, and ill presented, or flatly opposed to theory.

The modern reader will cite as objections against the foregoing coarse summary many individual masterpieces that do not support it: brilliant comparisons of theory with experiment by St. Venant, independent experiments of fundamental importance by Wertheim, Cauchy's marvellously clear mathematical apparatus for conceiving stress and arbitrarily large strains and rotations, theories of internal friction and plasticity proposed by Boltzmann, St. Venant, and others. If he is searching for antecedents of what has happened in the second half of the twentieth century, he is abundantly right in citing these and other achievements of the nineteenth while passing over the work of the ruck, but in that century's gross product of solid mechanics they are exceptions that prove rules.

In planning this volume on the mechanics of solids for the *Encyclopedia of Physics* I designed

1) To provide a treatise on experimental mechanics of solids that, not dominated by mathematical theory and not neglecting the work of the eighteenth and nineteenth centuries in favor of recent, more popular, and more costly forays, should be comparable in authority, breadth, and scholarship with LOVE's.
2) To provide treatises on basic, mathematical theory that would stand at the level of LOVE's while in their own, narrower scope supplanting it by compact and efficient development of fundamentals, making use of modern, incisive, yet elementary mathematics to weave together old and recent insights and achievements.
3) To illustrate the power of modern mathematical theory and modern experiment by articles on selected topics recently developed for their intellectual and practical importance, these two qualities being closer to each other than to some they may seem.

I encouraged the authors to meet the standard established by LOVE in just citation and temperate respect for the discoverers.

The reader will be able to form his own judgment of such success and failure as did accrue.

On the first head, experiment in general, the reader will find the treatise by Mr. BELL, filling all of Part 1. While it is not primarily a historical work, the historian S. G. BRUSH pronounced it in 1975 "the most important new publication by a single author" on the history of physics.

On the second head, the reader should not expect to find the basic ideas of solids treated *ab novo* or in isolation. The general and unified mechanics of EULER and CAUCHY, in which fluids, solids, and materials of other kinds are but instances, has come into its own in our day. No wise scientist now can afford to shut out solids when studying fluids or to forget the nature and peculiarities of fluids when studying solids. The two are but extreme examples in the class of systems comprised by mechanics. Articles in Parts 1 and 3 of Volume III of the *Encyclopedia: The Classical Field Theories* and *The Non-Linear Field Theories of Mechanics,* are cited so often by the authors writing in Volume VIa as to make it fatuous to deny that they provide the basic concepts, structures, and mathematical apparatus for the articles on theoretical mechanics of solids. In particular *The Non-Linear Field Theories* goes into such detail regarding large mechanical deformation as to allow most of the text in Volume VIa to concentrate upon small strain.

This much understood, we see that while Mr. BELL's volume provides, at last, a monument of exposition and scholarship on experiment, the articles by Messrs. GURTIN, CARLSON, FICHERA, NAGHDI, ANTMAN, and FISHER & LEITMAN, by Mrs. GEIRINGER, and by Mr. TING together provide a modern treatise on mathematical theories of the classical kinds. The survey of theories of elastic stability by Messrs. KNOPS & WILKES, now justly regarded as the standard reference for its field, necessarily considers deformations that need not be small.

Coming finally to application, in which theory and experiment complement one another, the reader will find major examples in the articles by Messrs. CHEN; NUNZIATO, WALSH, SCHULER and BARKER; and THURSTON. Many more topics of application might have been included. I regret that I could not secure articles about them. The most serious want is a survey of applications of linear elasticity to problems of intrinsic or applied interest that have arisen in this century and that illustrate the power of new mathematical analysis in dealing with special problems. A long article of that kind, a veritable treatise, was twice contracted and twice defaulted. Fortunately the gap thus left has been abundantly and expertly filled by Mr. VILLAGGIO, *Qualitative Methods in Elasticity,* Leyden, Noordhoff, 1977.

Baltimore, December, 1983 C. TRUESDELL

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The Linear Theory of Elasticity.

By

MORTON E. GURTIN.

With 18 Figures.

Dedicated to ELI STERNBERG.

A. Introduction.

1. Background. Nature of this treatise. Linear elasticity is one of the more successful theories of mathematical physics. Its pragmatic success in describing the small deformations of many materials is uncontested. The origins of the three-dimensional theory go back to the beginning of the 19th century and the derivation of the basic equations by CAUCHY, NAVIER, and POISSON. The theoretical development of the subject continued at a brisk pace until the early 20th century with the work of BELTRAMI, BETTI, BOUSSINESQ, KELVIN, KIRCHHOFF, LAMÉ, SAINT-VENANT, SOMIGLIANA, STOKES, and others. These authors established the basic theorems of the theory, namely compatibility, reciprocity, and uniqueness, and deduced important general solutions of the underlying field equations. In the 20th century the emphasis shifted to the solution of boundary-value problems, and the theory itself remained relatively dormant until the middle of the century when new results appeared concerning, among other things, Saint-Venant's principle, stress functions, variational principles, and uniqueness.

It is the purpose of this treatise to give an exhaustive presentation of the linear theory of elasticity.[1] Since this volume contains two articles by FICHERA concerning existence theorems, that subject will not be discussed here.

I have tried to maintain the level of rigor now customary in pure mathematics. However, in order to ease the burden on the reader, many theorems are stated with hypotheses more stringent than necessary.

Acknowledgement. I would like to acknowledge my debt to my friend and teacher, ELI STERNBERG, who showed me in his lectures[2] that it is possible to present the linear theory in a concise and rational form—a form palatable to both engineers and mathematicians. Portions of this treatise are based on STERNBERG's unpublished lecture notes; I have tried to indicate when such is the case. I would like to express my deep gratitude to D. CARLSON, G. FICHERA, R. HUILGOL, E. STERNBERG, and C. TRUESDELL for their valuable detailed criticisms of the manuscript. I would also like to thank G. BENTHIEN, W. A. DAY, J. ERICKSEN, R. KNOPS, M. OLIVER, G. DE LA PENHA, T. RALSTON, L. SOLOMON, E. WALSH, L. WHEELER, and W. WILLIAMS for valuable comments, and H. ZIEGLER for generously sending me a copy of

[1] Specific applications are not taken up in this article. They will be treated in a sequel by L. SOLOMON, *Some Classic Problems of Elasticity*, to appear in the Springer Tracts in Natural Philosophy.

[2] At Brown University in 1959–1961.

PRANGE's 1916 Habilitation Dissertation. Most of the historical research for this treatise was carried out at the Physical Sciences Library of Brown University; without the continued support and hospitality of the staff of that great library this research would not have been possible. Finally, let me express my gratitude to the U.S. National Science Foundation for their support through a research grant to Carnegie-Mellon University.

2. Terminology and general scheme of notation. I have departed radically from the customary notation in order to present the theory in what I believe to be a form most easily understood by someone not prejudiced by a past acquaintance with the subject. Direct notation, rather than cartesian or general coordinates, is utilized throughout. I do not use what is commonly called "dyadic notation"; most of the notions used, e.g. vector, linear transformation, tensor product, can be found in a modern text in linear algebra.

General scheme of notation.

Italic boldface minuscules $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{u}, \boldsymbol{v}$ ...: vectors and vector fields; $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{\xi}$, ...: points of space.

Italic boldface majuscules $\boldsymbol{A}, \boldsymbol{B}$, ...: (second-order) tensors or tensor fields.

Italic lightface letters A, a, α, Φ, ...: scalars or scalar fields.

C, K: fourth-order tensors.

Sans-serif boldface majuscules M, N ... (except C, K): four-tensors or fields with such values.

Sans-serif boldface minuscules $\mathsf{u}, \boldsymbol{\xi}$, ...: four-vectors or four-vector fields.

Italic lightface majuscules B, D, Σ, ...: regions in euclidean space.

$\mathscr{G}$: a group of second order tensors.

Script majuscules $\mathscr{C}, \mathscr{S}$, ... (except $\mathscr{G}$): surfaces in euclidean space.

Italic indices i, j, ...: tensorial indices with the range (1, 2, 3).

Greek indices α, β, ...: tensorial indices with the range (1, 2).

Index of frequently used symbols. Only symbols used frequently are listed. It has not been possible to adhere rigidly to these notations, so that sometimes within a single section these same letters are used for quantities other than those listed below.

Symbol	Name	Place of definition or first occurrence
$\boldsymbol{A}(\boldsymbol{m})$	Acoustic tensor	244
$\boldsymbol{A}$	Beltrami stress function	54
B	Body	14
C	Elasticity tensor	68
$D_{\boldsymbol{l}}$	Set of points of application of system $\boldsymbol{l}$ of concentrated loads	179
$\boldsymbol{E}$	Strain tensor	31
$\overset{\circ}{\boldsymbol{E}}$	Traceless part of strain tensor	78
$\overline{\boldsymbol{E}}(B)$	Mean strain	37
$\mathscr{E}$	Three-dimensional euclidean space	5
$\mathscr{E}^{(4)}$	$=\mathscr{E}\times(-\infty, \infty)=$ space-time	27

Symbol	Name	Place of definition or first occurrence
$\mathscr{G}_{\boldsymbol{x}}$	Symmetry group for the material at $\boldsymbol{x}$	70
K	Kinetic energy	64
$\mathbf{K}$	Compliance tensor	69
$\mathbf{M}$	Stress-momentum tensor	67
$\mathbf{0}$	Origin, zero vector, zero tensor	5, 6
P	Part of B	14
$\boldsymbol{Q}$	Orthogonal tensor	7
R	Plane region	154
$\boldsymbol{S}$	Stress tensor	45
$\mathring{\boldsymbol{S}}$	Traceless part of stress tensor	78
$\bar{\boldsymbol{S}}(B)$	Mean stress	61
$\mathscr{S}_1, \mathscr{S}_2$	Complementary subsets of ∂B	14
$U\{\boldsymbol{E}\}$	Strain energy	94
$\mathscr{U}$	Total energy	216
$\mathscr{V}$	Vector space associated with $\mathscr{E}$	5
$\mathscr{V}^{(4)}$	$=\mathscr{V}\times(-\infty, \infty)$	27
$\boldsymbol{W}$	Rotation tensor	31
$\mathscr{W}$	Singular surface	248
$\boldsymbol{a}$	Amplitude of wave	245, 254
$\boldsymbol{b}$	Body force	44
$\boldsymbol{c}$	Centroid of B	185
c	Speed of propagation, also the constant $16\pi\mu(1-\nu)$	246, 248
c_1	Irrotational velocity	213
c_2	Isochoric velocity	213
$\boldsymbol{e}_i$	Orthonormal basis	5
$\boldsymbol{f}$	Pseudo body force field	66
$\mathscr{f}$	System of forces	43
$\boldsymbol{i}$	$\sqrt{-1}$, also function with values $\boldsymbol{i}(t)=t$	65
k	Modulus of compression	74
$\boldsymbol{l}$	System of concentrated loads	179
$\boldsymbol{m}$	Direction of propagation	245, 248
$\boldsymbol{n}$	Outward unit normal vector on ∂B	14
p	Pressure	50
$\boldsymbol{p}$	Position vector from the origin $\mathbf{0}$	5
$\boldsymbol{p}_{\boldsymbol{c}}$	Position vector from the centroid $\boldsymbol{c}$	185
$\mathscr{p}$	Admissible process, elastic process	215
r	$=\lvert\boldsymbol{x}-\mathbf{0}\rvert$	21
$\boldsymbol{s}$	Surface traction	59
$\hat{\boldsymbol{s}}$	Prescribed surface traction	102
$\mathscr{s}$	Admissible state, elastic state	94, 95
$\mathscr{s}_{\boldsymbol{y}}[\boldsymbol{l}]$	Kelvin state corresponding to a concentrated load $\boldsymbol{l}$ at $\boldsymbol{y}$	174
$\mathscr{s}_{\boldsymbol{y}}^{i}$	Unit Kelvin state corresponding to the unit load $\boldsymbol{e}_i$ at $\boldsymbol{y}$	178
$\bar{\mathscr{s}}_{\boldsymbol{y}}^{ij}$	Unit doublet states at $\boldsymbol{y}$	178
$\mathring{\mathscr{s}}_{\boldsymbol{y}}$	Center of compression at $\boldsymbol{y}$	179
$\mathring{\mathscr{s}}_{\boldsymbol{y}}^{i}$	Center of rotation at $\boldsymbol{y}$ parallel to the x_i-axis	179

Symbol	Name	Place of definition or first occurrence
t	Time	24
$\boldsymbol{u}$	Displacement vector	31
$\hat{\boldsymbol{u}}$	Prescribed displacement on boundary	102
$\boldsymbol{u}_0$	Initial displacement	219
$\boldsymbol{v}_0$	Initial velocity	219
$v(B)$	Volume of B	35
$\boldsymbol{w}$	Rigid displacement	31
$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$	Points in space	2
x_i	Cartesian components of $\boldsymbol{x}$	5
z	Complex variable	159
$\Theta\{s\}$	Functional in Hellinger-Prange-Reissner principle	124
$\Lambda\{s\}$	Functional in Hu-Washizu principle	122
$\Sigma_\delta(\boldsymbol{y})$	Open ball with radius δ and center at $\boldsymbol{y}$	12
$\Phi\{s\}$	Functional in principle of minimum potential energy	111
$\Psi\{s\}$	Functional in principle of minimum complementary energy	112
β	Young's modulus	78
δ_{ij}	Kronecker's delta	5
$\delta v(B)$	Volume change	31
ε	Internal energy density	82
ε_{ijk}	Three-dimensional alternator	5
$\varepsilon_{\alpha\beta}$	Two-dimensional alternator	10
λ	Lamé modulus	76
μ	Shear modulus	76
μ_M	Maximum elastic modulus	85
μ_m	Minimum elastic modulus	85
ν	Poisson's ratio	78
$\boldsymbol{\xi}$	Point in space-time	27
ϱ	Density	43
φ	Scalar field in Boussinesq-Papkovitch-Neuber solution, Airy stress function, scalar field in Lamé solution	139, 157, 233
$\boldsymbol{\psi}$	Vector field in Boussinesq-Papkovitch-Neuber solution, vector field in Lamé solution	139, 233
$\boldsymbol{\omega}$	Rotation vector	31
$\mathbf{1}$	Unit tensor	6
sym	Symmetric part of a tensor	6
skw	Skew part of a tensor	6
tr	Trace of a tensor	6
$\otimes$	Tensor product of two vectors	7
∇	Gradient	10
$\nabla_{(4)}$	Gradient in space-time	28
$\hat{\nabla}$	Symmetric gradient	11
curl	Curl	11
div	Divergence	11
$\text{div}_{(4)}$	Divergence in space-time	28
Δ	Laplacian	11