Stochastic models, estimation, and control
VOLUME 2

PETER S. MAYBECK

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Preface

As was true of Volume 1, the purpose of this book is twofold. First, it attempts to develop a thorough understanding of the fundamental concepts incorporated in stochastic processes, estimation, and control. Second, and of equal importance, it provides experience and insights into applying the theory to realistic practical problems. Basically, it investigates the theory and derives from it the tools required to reach the ultimate objective of systematically generating effective designs for estimators and stochastic controllers for operational implementation.

Perhaps most importantly, the entire text follows the basic principles of Volume 1 and concentrates on presenting material in the most lucid, best motivated, and most easily grasped manner. It is oriented toward an engineer or an engineering student, and it is intended both to be a textbook from which a reader can *learn* about estimation and stochastic control and to provide a good reference source for those who are deeply immersed in these areas. As a result, considerable effort is expended to provide graphical representations, physical interpretations and justifications, geometrical insights, and practical implications of important concepts, as well as precise and mathematically rigorous development of ideas. With an eye to practicality and eventual implementation of algorithms in a digital computer, emphasis is maintained on the case of continuous-time dynamic systems with sampled-data measurements available; nevertheless, corresponding results for discrete-time dynamics or for continuous-time measurements are also presented. These algorithms are developed in detail, to the point where the various design trade-offs and performance evaluations involved in achieving an efficient, practical configuration can be understood. Many examples and problems are used throughout the text to aid comprehension of important concepts. Furthermore, there is an extensive set of references in each chapter to allow pursuit of ideas in the open literature once an understanding of both theoretical concepts and practical implementation issues has been established through the text.

This volume builds upon the foundations set in Volume 1. The seven chapters of that volume yielded linear stochastic system models driven by white Gaussian noises and the optimal Kalman filter based upon models of that form. In this volume, Chapters 8–10 extend these ideas to consider optimal smoothing in addition to filtering, compensation of linear model inadequacies while exploiting the basic insights of linear filtering (including an initial study of the important extended Kalman filter algorithm), and adaptive estimation based upon linear models in which uncertain parameters are embedded. Subsequently, Chapter 11 properly develops nonlinear stochastic system models, which then form the basis for the design of practical nonlinear estimation algorithms in Chapter 12.

This book forms a self-contained set with Volume 1, and together with Volume 3 on stochastic control, can provide a fundamental source for studying stochastic models, estimation, and control. In fact, they are an outgrowth of a three-quarter sequence of graduate courses taught at the Air Force Institute of Technology; and thus the text and problems have received thorough class testing. Students had previously taken a basic course in applied probability theory, and many had also taken a first control theory course, linear algebra, and linear system theory; but the required aspects of these disciplines have also been developed in Volume 1. The reader is assumed to have been exposed to advanced calculus, differential equations, and some vector and matrix analysis on an engineering level. Any more advanced mathematical concepts are developed within the text itself, requiring only a willingness on the part of the reader to deal with new means of conceiving a problem and its solution. Although the mathematics becomes relatively sophisticated at times, efforts are made to motivate the need for, and to stress the underlying basis of, this sophistication.

The author wishes to express his gratitude to the many students who have contributed significantly to the writing of this book through their feedback to me—in the form of suggestions, questions, encouragement, and their own personal growth. I regard it as one of God's many blessings that I have had the privilege to interact with these individuals and to contribute to their growth. The stimulation of technical discussions and association with Professors Michael Athans, John Deyst, Nils Sandell, Wallace Vander Velde, William Widnall, and Alan Willsky of the Massachusetts Institute of Technology, Professor David Kleinman of the University of Connecticut, and Professors Jurgen Gobien, James Negro, J. B. Peterson, and Stanley Robinson of the Air Force Institute of Technology has also had a profound effect on this work. I deeply appreciate the continued support provided by Dr. Robert Fontana, Chairman of the Department of Electrical Engineering at AFIT, and the painstaking care with which many of my associates have reviewed the manuscript. Finally, I wish to thank my wife, Beverly, and my children, Kristen and Keryn, without whose constant love and support this effort could not have been fruitful.

Notation

Vectors, Matrices

Scalars are denoted by upper or lower case letters in italic type.

Vectors are denoted by lower case letters in boldface type, as the vector $\mathbf{x}$ made up of components x_i.

Matrices are denoted by upper case letters in boldface type, as the matrix $\mathbf{A}$ made up of elements A_{ij} (ith row, jth column).

Random Vectors (Stochastic Processes), Realizations (Samples), and Dummy Variables

Random vectors are set in boldface sans serif type, as $\mathsf{x}(\cdot)$ or frequently just as x made up of scalar components x_i; $\mathsf{x}(\cdot)$ is a mapping from the sample space Ω into real Euclidean n-space R^n: for each $\omega_k \in \Omega$, $\mathsf{x}(\omega_k) \in R^n$.

Realizations of the random vector are set in boldface roman type, as $\mathbf{x}$: $\mathsf{x}(\omega_k) = \mathbf{x}$.

Dummy variables (for arguments of density or distribution functions, integrations, etc.) are denoted by the equivalent Greek letter, such as $\boldsymbol{\xi}$ being associated with x: e.g., the density function $f_{\mathsf{x}}(\boldsymbol{\xi})$. The correspondences are $(\mathsf{x}, \boldsymbol{\xi})$, $(\mathsf{y}, \boldsymbol{\rho})$, $(\mathsf{z}, \boldsymbol{\zeta})$, $(\mathsf{Z}, \mathscr{Z})$.

Stochastic processes are set in boldface sans serif type, just as random vectors are. The n-vector stochastic process $\mathsf{x}(\cdot,\cdot)$ is a mapping from the product space $T \times \Omega$ into R^n, where T is some time set of interest: for each $t_j \in T$ and $\omega_k \in \Omega$, $\mathsf{x}(t_j, \omega_k) \in R^n$. Moreover, for each $t_j \in T$, $\mathsf{x}(t_j, \cdot)$ is a random vector, and for each $\omega_k \in \Omega$, $\mathsf{x}(\cdot, \omega_k)$ can be thought of as a particular time function and is called a *sample* out of the process. In analogy with random vector realizations, such samples are set in boldface roman type: $\mathsf{x}(\cdot, \omega_k) = \mathbf{x}(\cdot)$ and $\mathsf{x}(t_j, \omega_k) = \mathbf{x}(t_j)$. Often the second argument of a stochastic process is suppressed: $\mathsf{x}(t, \cdot)$ is often written as $\mathsf{x}(t)$, and this stochastic process evaluated at time t is to be distinguished from a process sample $\mathbf{x}(t)$ at that same time.

Subscripts

a:	augmented	n:	nominal
b:	backward running	ss:	steady state
c:	continuous-time	t:	truth model
d:	discrete-time	0:	initial time
f:	final time; or filter (shaping filter)		

Superscripts

$^{\mathrm{T}}$:	transpose (matrix)	$^{\#}$:	pseudoinverse
*:	complex conjugate transpose; or transformed coordinates	$\hat{}$:	estimate
$^{-1}$:	inverse (matrix)	$\bar{}$:	Fourier transform; or steady state solution

Matrix and Vector Relationships

$\mathbf{A} > \mathbf{0}$: $\mathbf{A}$ is positive definite.
$\mathbf{A} \geq \mathbf{0}$: $\mathbf{A}$ is positive semidefinite.
$\mathbf{x} \leq \mathbf{a}$: componentwise, $x_1 \leq a_1$, $x_2 \leq a_2, \ldots,$ and $x_n \leq a_n$.

Commonly Used
Abbreviations and Symbols

$E\{\cdot\}$	expectation	w.p.1	with probability of one
$E\{\cdot\vert\cdot\}$	conditional expectation	$\vert\cdot\vert$	determinant of
exp($\cdot$)	exponential	$\Vert\cdot\Vert$	norm of
lim.	limit	$\sqrt{\cdot}$	matrix square root of (see Volume 1)
l.i.m.	limit in mean (square)	$\in$	element of
ln($\cdot$)	natural log	$\subset$	subset of
m.s.	mean square	$\{\cdot\}$	set of; such as $\{\mathbf{x} \in \mathbf{X} : \mathbf{x} \leq \mathbf{a}\}$, i.e., the set of $\mathbf{x} \in \mathbf{X}$ such that $x_i \leq a_i$ for all i
max.	maximum		
min.	minimum		
R^n	Euclidean n-space		
sgn($\cdot$)	signum (sign of)		
tr($\cdot$)	trace		

List of symbols and pages where they are defined or first used

$\mathbf{A}$	14; 78; 137	$\mathbf{A}_{\mathrm{TS}}$	220; 225
$\mathbf{A}_{\mathrm{GS}}$	222; 226	$\mathbf{a}$	75

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Chapter 9 Compensation of linear model inadequacies

Chapter 10 Parameter uncertainties and adaptive estimation

Chapter 11 Nonlinear stochastic system models

Chapter 12 Nonlinear estimation

VOLUME 3

CHAPTER 8

Optimal smoothing

8.1 INTRODUCTION

In the previous chapters, we have considered linear system models and optimal filtering, the optimal estimation of the state at time t_i, $\mathbf{x}(t_i, \omega_j) = \mathbf{x}(t_i)$, based upon knowledge of all measurements taken up to time t_i:

$$\mathbf{z}(t_1, \omega_j) = \mathbf{z}_1, \quad \mathbf{z}(t_2, \omega_j) = \mathbf{z}_2, \quad \ldots, \quad \mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$$

or equivalently, $\mathbf{Z}(t_i, \omega_j) = \mathbf{Z}_i$. We have actually considered optimal prediction as well in attempting to estimate $\mathbf{x}(t_i)$ based on knowledge of $\mathbf{Z}(t_{i-1}, \omega_j) = \mathbf{Z}_{i-1}$. Under our assumptions, the optimal estimate of $\mathbf{x}(t_i)$, based on knowledge of available measurement information, has been the conditional expectation of $\mathbf{x}(t_i)$, conditioned on that information:

$$\hat{\mathbf{x}}(t_i^+) = E\{\mathbf{x}(t_i) | \mathbf{Z}(t_i, \omega_j) = \mathbf{Z}_i\} \tag{8-1}$$

$$\hat{\mathbf{x}}(t_i^-) = E\{\mathbf{x}(t_i) | \mathbf{Z}(t_{i-1}, \omega_j) = \mathbf{Z}_{i-1}\} \tag{8-2}$$

In fact, these values were shown to be optimal with respect to many different criteria.

The Kalman filter, or square root implementation of the same estimator, provides the best estimate of $\mathbf{x}(t_i)$ based on all measurements through time t_i in a recursive manner, and it is thus ideally suited to real-time computations. However, if one were willing (or able) to wait until after time t_i to generate an optimal estimate of $\mathbf{x}(t_i)$, then a *better* estimate than the $\hat{\mathbf{x}}(t_i^+)$ provided by the Kalman filter could be produced in most cases [6, 7, 9, 23, 28]. The additional information contained in the measurements taken after time t_i can be exploited to provide this improvement in estimation accuracy. The *optimal smoothed estimate* [36] (again under many criteria) is

$$\hat{\mathbf{x}}(t_i/t_j) = E\{\mathbf{x}(t_i) | \mathbf{Z}(t_j, \omega_k) = \mathbf{Z}_j\}, \qquad j > i \tag{8-3}$$

and the subject of optimal smoothing is concerned with developing efficient, practical algorithms for calculating this estimate.

Section 8.2 formulates the smoothing problem and presents a conceptual approach to smoothing of combining the outputs of a filter running forward from initial time t_0 to the current time t_i, and a separate filter running backward from terminal time t_f to t_i. Three useful classes of smoothing problems, characterized by the manner in which t_i and t_j can vary in (8-3), are presented in Section 8.3, and then discussed individually in the ensuing three sections.

8.2 BASIC STRUCTURE

Explicit equations for various forms of optimal smoothers are generally quite complicated. However, the basic smoothing concept and underlying structure can be discerned readily by dividing the estimation problem into two parts, one involving the past and present measurements and the other based on future measurements alone, and combining the results.

Consider a discrete-time model (possibly "equivalent discrete"):

$$\mathbf{x}(t_{i+1}) = \mathbf{\Phi}(t_{i+1}, t_i)\mathbf{x}(t_i) + \mathbf{B}_d(t_i)\mathbf{u}(t_i) + \mathbf{G}_d(t_i)\mathbf{w}_d(t_i) \tag{8-4}$$

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \tag{8-5}$$

with the usual assumptions on $\mathbf{x}(t_0)$, $\mathbf{w}_d(\cdot,\cdot)$, and $\mathbf{v}(\cdot,\cdot)$: Gaussian and independent of each other, initial conditions with mean $\hat{\mathbf{x}}_0$ and covariance $\mathbf{P}_0$, white and zero-mean processes of strengths $\mathbf{Q}_d(t_i)$ and $\mathbf{R}(t_i)$, respectively, for all times of interest. Now assume we are trying to estimate $\mathbf{x}(t_i)$ from measurement data through time t_j, with $j > i$. Put all of the measurements up through time t_i into a single composite vector $\mathbf{Z}(t_i)$, or perhaps more explicitly $\mathbf{Z}(t_1, t_i)$, denoting the fact that its partitions are $\mathbf{z}(t_1)$, $\mathbf{z}(t_2), \ldots, \mathbf{z}(t_i)$. Similarly, put all "future" measurements, $\mathbf{z}(t_{i+1}), \mathbf{z}(t_{i+2}), \ldots, \mathbf{z}(t_j)$, into a single composite vector $\mathbf{Z}(t_{i+1}, t_j)$. Conceptually, a three-part procedure can now be employed to estimate $\mathbf{x}(t_i)$:

(1) Calculate

$$\hat{\mathbf{x}}(t_i^+) = E\{\mathbf{x}(t_i) | \mathbf{Z}(t_1, t_i) = \mathbf{Z}_{1,i}\} \tag{8-6}$$

by means of a filter running forward in time from time t_0 to time t_i. A priori information about $\mathbf{x}(t_0)$ is used to initialize this filter.

(2) Independently, calculate

$$\hat{\mathbf{x}}_b(t_i^-) = E\{\mathbf{x}(t_i) | \mathbf{Z}(t_{i+1}, t_j) = \mathbf{Z}_{(i+1),j}\} \tag{8-7}$$

by means of a filter that is run backwards in time from time t_j to time t_{i+1}, plus a one-step "prediction" backward to time t_i. The notation $\hat{\mathbf{x}}_b(t_i^-)$ is meant to denote the estimate of $\mathbf{x}(t_i)$ provided by the backward-running filter (thus the subscript b), just before the measurement at time t_i is incorporated (thus the minus superscript on t_i^-). Note that time t_i^- is to the right of t_i^+ on a real-time scale for the backward filter, as shown in Fig. 8.1, since minus and plus denote before and after measurement incorporation, respectively. The "initial" condition for the backward-running filter is established by viewing $\mathbf{x}(t_j)$ as a random vector about which you have no a priori statistical information, i.e., $\mathbf{P}_b^{-1}(t_j^-) = \mathbf{0}$. Thus, an inverse-covariance formulation is appropriate for the backward filter. (This will be developed further in Section 8.4.)

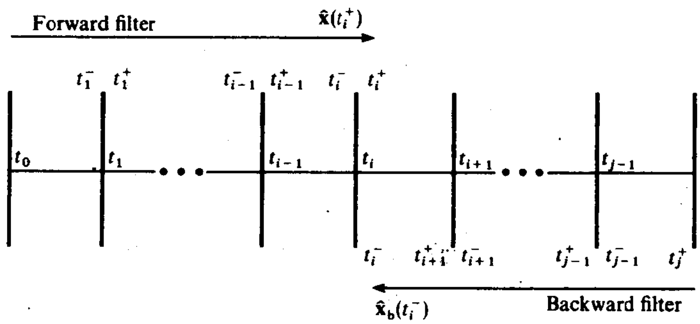

FIG. 8.1 Forward and backward filter operation.

(3) The smoothed estimate of $\mathbf{x}(t_i)$, $\hat{\mathbf{x}}(t_i/t_j)$ as defined in (8-3), is generated by optimally combining the value of $\hat{\mathbf{x}}(t_i^+)$ from the forward filter (incorporating initial condition information about $\mathbf{x}(t_0)$ and measurement information from $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_i$) and $\hat{\mathbf{x}}_b(t_i^-)$ from the backward filter (incorporating measurement information from $\mathbf{z}_{i+1}, \mathbf{z}_{i+2}, \ldots, \mathbf{z}_j$). This combination is accomplished by viewing $\hat{\mathbf{x}}(t_i^+)$ and $\hat{\mathbf{x}}_b(t_i^-)$ as two separate "observations" of $\mathbf{x}(t_i)$ and assigning relative weighting according to the confidence you have in the precision of each, indicated by $\mathbf{P}(t_i^+)$ and $\mathbf{P}_b(t_i^-)$, respectively. Another way of thinking of this process is to consider the backward filter output $\hat{\mathbf{x}}_b(t_i^-)$ as providing an additional "measurement" with which to update the forward filter. Note that we choose to process $\mathbf{z}(t_i, \omega_k) = \mathbf{z}_i$ in the forward filter; we could just as easily have chosen to process it in the backward filter instead, as long as this data does not enter into both filters and thus be counted twice in the smoothed estimate.

8.3 THREE CLASSES OF SMOOTHING PROBLEMS

There are many different classes of smoothing problems, each being determined by the manner in which the time parameters t_i and t_j are allowed to vary in the desired smoothed estimate $\hat{\mathbf{x}}(t_i/t_j)$. However, there are three classes of particular interest because of their applicability to realistic problems, namely, fixed-interval, fixed-point, and fixed-lag smoothing problems [10, 23, 24, 27, 32, 36].

To describe *fixed-interval smoothing*, let an experiment (system operation, mission, etc.) be conducted, and let measurement data be collected over the interval from initial time t_0 to final time t_f, $[t_0, t_f]$. After all of the data has been collected, it is desired to obtain the optimal estimate of $\mathbf{x}(t_i)$ for all time $t_i \in [t_0, t_f]$, based on all measurements taken in the interval. Offline computations are thus inherently involved in generating the optimal fixed-interval smoothed estimate,

$$\hat{\mathbf{x}}(t_i/t_f) = E\{\mathbf{x}(t_i)|\mathbf{Z}(t_f) = \mathbf{Z}_f\}$$
$$t_i = t_0, t_1, \ldots, t_f; \qquad t_f = \text{fixed final time} \tag{8-8}$$

Figure 8.2a represents these calculations schematically. This estimation technique is used for post-experiment data reduction to obtain refined state estimates

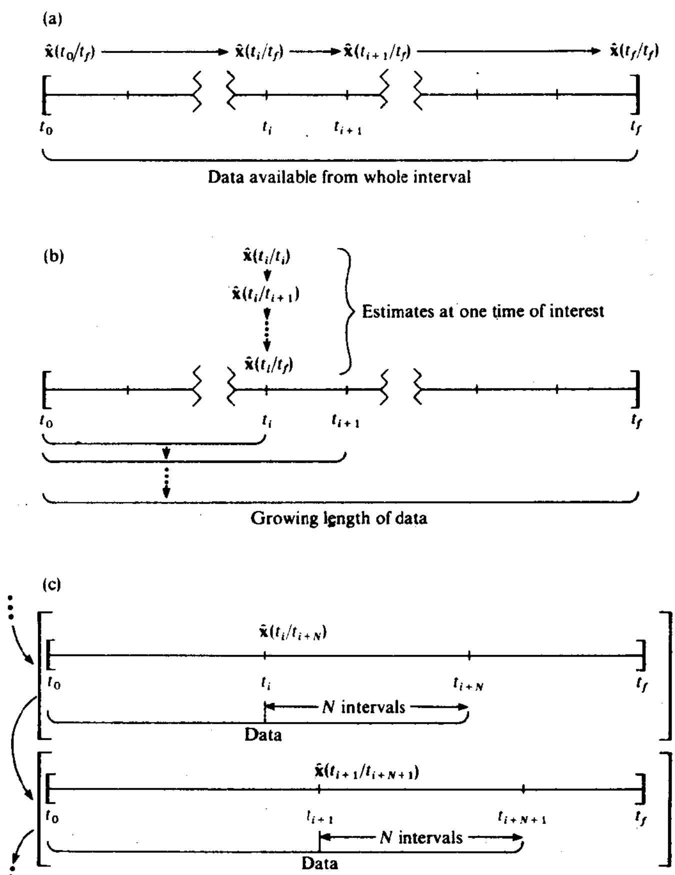

FIG. 8.2 Three types of smoothers: (a) fixed-interval smoothing, (b) fixed-point smoothing, (c) fixed-lag smoothing.

of better quality than that provided by online filters. It is also possible to use fixed-interval smoothing to estimate values of control inputs as well as states, to assess whether the "deterministic" controls were actually of commanded magnitude. A specific example would be post-flight analysis of a missile, generating smoothed estimates of both trajectory parameters (states) and thrust actually produced by the rocket motors for that flight (controls).

To consider *fixed-point smoothing*, let there be a certain point (or points) in time at which the value of the system state is considered critical. For example, conditions at engine burnout time are critical to rocket booster problems. Thus, one would desire an estimate of $\mathbf{x}(t_i)$ for fixed t_i, conditioned on more and more data as measurements become available in real time:

$$\begin{gathered}\hat{\mathbf{x}}(t_i/t_j) = E\{\mathbf{x}(t_i)|\mathbf{Z}(t_j) = \mathbf{Z}_j\} \\ t_i \text{ fixed}; \qquad t_j = t_i, t_{i+1}, \ldots, t_f\end{gathered} \tag{8-9}$$

This is the optimal fixed-point smoothed estimate, as depicted in Fig. 8.2b.

Finally, let measurements be taken, but assume that it is admissible for your application to generate an optimal estimate of $\mathbf{x}(t_i)$, not at time t_i, but at time t_{i+N}, where N is a fixed integer. Thus, to estimate $\mathbf{x}(t_i)$, you have available not only the measurements

$$\mathbf{z}(t_1, \omega_k) = \mathbf{z}_1, \quad \mathbf{z}(t_2, \omega_k) = \mathbf{z}_2, \quad \ldots, \quad \mathbf{z}(t_i, \omega_k) = \mathbf{z}_i$$

but also the N additional measurements

$$\mathbf{z}(t_{i+1}, \omega_k) = \mathbf{z}_{i+1}, \quad \ldots, \quad \mathbf{z}(t_{i+N}, \omega_k) = \mathbf{z}_{i+N}$$

and you are willing to delay the computation of the estimate of $\mathbf{x}(t_i)$ until t_{i+N} to take advantage of the additional information in these N measurements. We wish to generate the optimal *fixed-lag smoothed estimate*,

$$\begin{gathered}\hat{\mathbf{x}}(t_i/t_{i+N}) = E\{\mathbf{x}(t_i)|\mathbf{Z}(t_{i+N}) = \mathbf{Z}_{i+N}\} \\ t_i = t_0, t_1, \ldots, t_{f-N}; \qquad N = \text{fixed integer}\end{gathered} \tag{8-10}$$

Such an estimator is depicted in Fig. 8.2c and is particularly applicable to communications and telemetry data reduction.

8.4 FIXED-INTERVAL SMOOTHING

To develop the fixed-interval smoother, we shall exploit the work of Fraser [6, 7], who first showed it to be just a suitable combination of two optimal filters. Let the *forward filter* recursively produce a state estimate $\hat{\mathbf{x}}(t_k^-)$ and error covariance $\mathbf{P}(t_k^-)$ before incorporation of measurement $\mathbf{z}_k$, and $\hat{\mathbf{x}}(t_k^+)$ and $\mathbf{P}(t_k^+)$ after incorporation, for $k = 1, 2, \ldots, i$. Notationally, let $\hat{\mathbf{x}}_b(t_k^-)$ and $\mathbf{P}_b(t_k^-)$ denote the state estimate and error covariance before incorporating measurement $\mathbf{z}_k$ into the *backward filter*, and let $\hat{\mathbf{x}}_b(t_k^+)$ and $\mathbf{P}_b(t_k^+)$ be analogous