Mixed Boundary Value Problems in Potential Theory

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BY

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PREFACE

In the spring of 1959 the author gave a series of lectures on mixed boundary value problems in mathematical physics to a group of graduate students and faculty members of Duke University where he was engaged in the Mathematics Department on research on partial differential equations under Contract AF 18(600) 1341. The material that was covered in these lectures forms the basis of the present book although it has been drastically revised in view of more recent work on the subject and in the light of the author's subsequent experience in giving courses of lectures in the Mathematical Institute of the Polish Academy of Sciences, Warsaw, in the autumn of 1959, in the University of Montreal in the summer of 1961 (as part of the Summer Seminar of the Fifth Canadian Mathematical Congress); shorter courses on the material which now forms Chapters IV, V were given in the Mathematics Department, North Carolina State University in the springs of 1962, 1963 respectively, and in the University of Zagreb in the spring of 1964.

The book surveys recent work on mixed boundary value problems in potential theory and on the mathematical tools devised for their solution; the original lectures attempted to discuss diffraction theory but it was felt that diffraction problems should be the subject of another book — written by someone more expert in this field than the present author.

An account is given of the ways in which mixed boundary value problems arise in potential theory and of the mathematical techniques which have been developed in recent years to solve such problems. Although the earliest papers are due to Weber (1873) and Beltrami (1881) it is only since 1945 that there have been attempts to develop the theory of dual integral equations (Chapter IV), dual series equations (Chapter V) and triple relations (Chapter VI); this work was initiated by the publication in 1937 of Titchmarsh's *An Introduction to the Theory of Fourier Integrals* which included the first systematic treatment of dual integral equations, and in 1938 of Miss Busbridge's paper extending the range of validity of Titchmarsh's solution. The systematic treatment of integral representations of harmonic functions (Chapter VII) is much older but it is only in recent years, due to the work of A. E. Green, A. E. Heins and W. D. Collins that it has been applied extensively to the solution of mixed boundary value problems.

No account is given here of the methods based on the theory of Cauchy integrals, developed by N. I. Muskhelishvili and his school, for the solution of mixed problems in the plane. These methods have played a decisive role in the development of potential theory and of the mathematical theory of elasticity but there is no need to describe them here since they are so clearly expounded in Muskhelishvili's own book (Muskhelishvili, 1953).

The application to physics of the methods developed in Chapters IV–VII is illustrated in Chapter VIII where these techniques are applied to the consideration of some problems in electrostatics. This has the advantage that the physical situation is simple but the procedures are easily extended to physically more complicated situations such as arise, for instance, in the theory of elasticity. A reader interested in this kind of application might consult the forthcoming book, Sneddon and Lowengrub (1966), where these methods are used in the solution of crack problems in the classical theory of elasticity. Although the boundary value problems discussed here are taken entirely from potential theory the methods developed in Chapters IV–VII are basic to one of the approaches to the study of diffraction problems.

The book is written primarily for students of applied mathematics, physics and engineering. Some knowledge of transform theory and of the special functions of mathematical physics is desirable but the necessary results involving Bessel functions, Jacobi polynomials, Legendre functions, integral equations and fractional integration are discussed in Chapter II. If only pure mathematicians were being catered for, we should have to exercise a good deal more care in formulating the proofs of theorems and to explore in considerably more depth numerous technical points which have been passed over lightly; our aim has been to make these powerful analytical tools available to as wide a class of reader as possible, and this can only be done by accepting standards of rigour which fall short of those which are current in modern abstract mathematics.

It has been my great pleasure to have talked about mixed boundary value problems with most of the authors whose names appear in the bibliography. Some have been my students, some my colleagues or collaborators, some I have heard lecture and with some I have had private conversations or correspondence; I like to think of all of them as my friends. In the preparation of the reports based on the lectures I was assisted by Dr. F. J. Lockett and Dr. M. Lowengrub. I am indebted also to Dr. J. Burlak, Dr. M. Lowengrub, Dr. R. P. Srivastav, Dr. M. P. Stallybrass and Miss Alison Baxter for their comments on the manuscript of the present version and for their help in reading the proofs. I am grateful also to Professor W. D. Collins for reading Chapter VII at the proof stage and for his advice on how it could be improved. My thanks are due also to Miss Jean Melville for her assistance in the preparation of the indices.

Finally I wish to express my gratitude to Dr. J. J. Gergen who invited me to Duke University in the first instance, who did so much to make my first and subsequent visits there a pleasant and intellectually stimulating experience, and who has at all stages encouraged me to complete the present book.

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The University of Glasgow
Scotland
28 February, 1966

CONTENTS

CHAPTER I

THE OCCURRENCE OF MIXED BOUNDARY VALUE PROBLEMS IN POTENTIAL THEORY

In this chapter we shall discuss briefly some of the mixed boundary value problems which arise in mathematical physics. We use the term "mixed" boundary value problem to distinguish this type of problem from the "uniform" problems of Dirichlet and Neumann. It will be recalled that a problem in potential theory is called a Dirichlet problem if the potential function whose form inside a region S is to be determined is prescribed at each point of ∂S, the boundary of S, and a Neumann problem if it is the normal derivative of the potential function which is prescribed on ∂S—not the function itself. In potential theory a typical problem of mixed kind would be one in which the potential function is prescribed over a part of the boundary, and its normal derivative is prescribed over the remaining part. In another kind of problem the potential function is prescribed over part of the boundary and a linear combination of the function and its normal derivative is prescribed over the remaining part.

1.1. Electrostatic Problems

One of the simplest problems which we can conceive in electrostatics is that of calculating the electrostatic potential of a circular disk which is charged

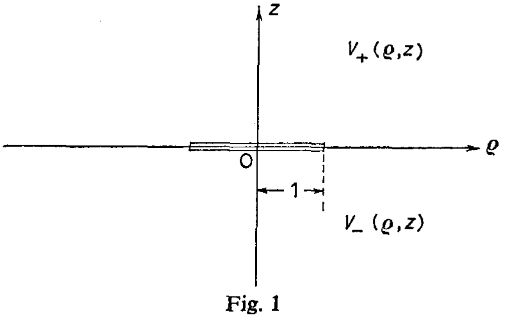

Fig. 1

to a prescribed potential. There is no loss of generality if we take our unit of length to be the radius of the disk. If we use cylindrical coordinates (ρ,ϑ,z) we may therefore take the disk to be $\rho \leqq 1$, $z = 0$.

If we denote by V_+ the potential in the half-space $z \geqq 0$, and by V_-, the potential in the half-space $z \leqq 0$, we see that V_+ and V_- must satisfy Laplace's equation, i.e.

$$\Delta V_+ = 0, \quad z > 0; \quad \Delta V_- = 0, \quad z < 0, \tag{1.1.1}$$

where, in cylindrical coordinates, the Laplacian operator Δ takes the form

$$\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \vartheta^2} + \frac{\partial^2}{\partial z^2}. \tag{1.1.2}$$

Further, these harmonic functions V_+ and V_- must satisfy the continuity conditions

$$V_+ = V_-, \text{ on } z = 0, \tag{1.1.3}$$

$$\frac{\partial V_+}{\partial z} - \frac{\partial V_-}{\partial z} = -4\pi\sigma(\rho,\vartheta), \text{ on } z = 0, \tag{1.1.4}$$

where $\sigma(\rho,\vartheta)$ denotes the surface charge density on the plane $z = 0$. Finally it is assumed that, in the absence of an external field

$$\left.\begin{aligned} V_+ \to 0 \text{ as } r \to \infty, \ z > 0 \\ V_- \to 0 \text{ as } r \to \infty, \ z < 0 \end{aligned}\right\} \tag{1.1.5}$$

where $r^2 = \rho^2 + z^2$.

If, therefore, we prescribe the potential over the surface of the disk to be $f(\rho,\vartheta)$, we have

$$V_+ = V_- = f(\rho,\vartheta), \quad z = 0, \quad \rho \leqq 1, \tag{1.1.6}$$

and if we make use of the fact that the surface charge density is zero outside the disk, we have

$$\frac{\partial V_+}{\partial z} - \frac{\partial V_-}{\partial z} = 0, \qquad z = 0, \qquad \rho > 1. \tag{1.1.7}$$

The problem of determining the electric field due to a disk charged to a known potential $f(\rho,\vartheta)$ is therefore solved if we can solve the equations (1.1.1) subject to the boundary conditions (1.1.5–7). If we wish to calculate the density of electric charge on the disk, we can then make use of equation (1.1.4). It is a simple matter to show that we can replace the problem of determining the two functions V_+, V_-, each defined in the relevant half-space, by that of finding a single potential function V, in $z > 0$, satisfying the boundary conditions

$$\left.\begin{aligned} V &= f(\rho,\vartheta), \;\; \rho \leqq 1, \;\; 0 \leqq \vartheta < 2\pi, \\ \frac{\partial V}{\partial z} &= 0, \qquad\quad \rho > 1, \;\; 0 \leqq \vartheta < 2\pi, \end{aligned}\right\} \tag{1.1.8}$$

on $z = 0$ (cf. §3.1 below), and vanishing as $r \to \infty$ through positive values of z.

In this way we reduce the problem of the electrified disk to that of solving a mixed boundary value problem for the positive half-space $z \geqq 0$, the mixed conditions being prescribed on the boundary $z = 0$. The same problem can, however, be reduced to a Dirichlet problem in the following way: we begin by considering the electric field in the space bounded by two concentric spheroids S_1 and S_2, it being assumed that the potential function V assumes prescribed

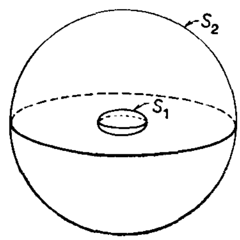

Fig. 2

values on the surface S_1 and is zero on the surface S_2. We then consider the limiting case in which the spheroid S_1 degenerates into a circular disk and the spheroid S_2 becomes a sphere of very large radius. In the simplest case this is indeed a useful way in which to proceed but the procedure is one which cannot easily be generalized to more complicated situations, such as those involving two circular disks, and, since it involves the use of a not too familiar system of orthogonal curvilinear coordinates, the results are often presented in a form which makes numerical calculation difficult (cf. §3.3. below).

The mixed boundary value problem posed by the equations (1.1.8) is probably the simplest one of its kind and, for that reason, it is regarded as the "classical" mixed boundary value problem.

Several generalizations immediately suggest themselves. We can think of the electrified disk as being enclosed within a concentric circular cylinder whose axis is normal to the plane of the disk and which is itself connected to ground. Because of the occurrence of finite boundaries we should expect this to be a much more difficult problem to solve than the previous one and in fact it proves to be so.

Suppose that the potential in the upper half of the space bounded by the cylinder is denoted by V_+ and in the lower half by V_-. Then the function V_+

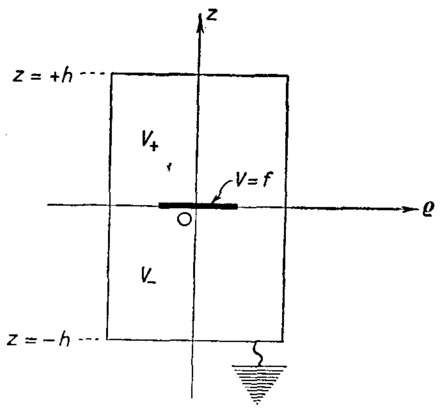

Fig. 3

must satisfy Laplace's equation in the region $0 \leqq z \leqq h$, $\rho \leqq a$ if the cylinder has radius a and height $2h$, and V_- must satisfy Laplace's equation in the region $-h \leqq z \leqq 0, \rho \leqq a$. The boundary conditions satisfied by these functions are easily seen to be

$$V_+ = V_- = f(\rho,\vartheta), \quad z = 0, \quad \rho \leqq 1, \tag{1.1.9}$$

$$\frac{\partial V_+}{\partial z} = \frac{\partial V_-}{\partial z}, \quad z = 0, \quad 1 < \rho \leqq a, \tag{1.1.10}$$

$$V_+ = 0, \text{ on } z = +h, \ \rho \leqq a, \text{ and on } \rho = a, \quad 0 \leqq z \leqq h, \tag{1.1.11}$$

$$V_- = 0, \text{ on } z = -h, \ \rho \leqq a, \text{ and on } \rho = a, \quad -h \leqq z \leqq 0. \tag{1.1.12}$$

By a method similar to that used in the case of an electrified disk in an infinite space we can reduce this problem to that of determining a function V, which is harmonic in the region $0 \leqq \rho < a$, $0 \leqq z < h$, and which satisfies the mixed boundary conditions

$$\left.\begin{aligned} V &= f(\rho,\vartheta), \quad z = 0, \quad 0 \leqq \rho \leqq 1 \\ \frac{\partial V}{\partial z} &= 0, \qquad z = 0, \quad 1 < \rho < a \\ V &= 0, \qquad \text{on } z = h, \text{ and on } \rho = a. \end{aligned}\right\} \tag{1.1.13}$$

Another obvious generalization of the problem of the electrified disk is the problem of determining the electrostatic field due to two parallel coaxial disks of equal radius (which may be taken as the unit of length) placed with their centres on the z-axis. Such an arrangement is shown in Fig. 4. If the potentials in the regions $z \leqq -\delta$, $-\delta \leqq z \leqq \delta$, $z \geqq +\delta$ are denoted respectively by V_-,

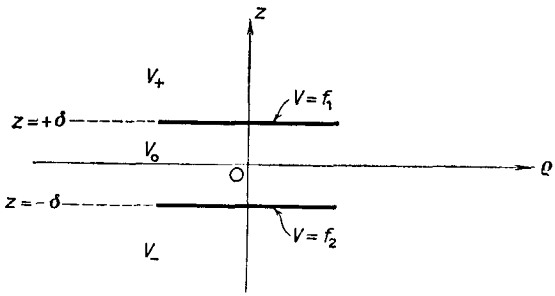

Fig. 4

V_0, V_+ and if the upper disk is charged to a prescribed potential $f_1(\rho,\vartheta)$ and the lower one to a potential $f_2(\rho,\vartheta)$ then we find that the three potential functions must satisfy the boundary conditions

$$V_+ = V_0 = f_1, \; 0 \leqq \rho < 1, \quad \frac{\partial V_+}{\partial z} = \frac{\partial V_0}{\partial z}, \; \rho > 1, \tag{1.1.14}$$

on the plane $z = +\delta$, and the boundary conditions

$$V_- = V_0 = f_2, \; 0 \leqq \rho < 1, \quad \frac{\partial V_0}{\partial z} = \frac{\partial V_-}{\partial z}, \; \rho > 1, \tag{1.1.15}$$

on the plane $z = -\delta$, and the further condition that each of these functions must tend to zero as $r \to \infty$. This problem is a classical one, a solution in a simple case having been derived by Riemann (1855).

We get another set of problems by replacing the circular disk $0 \leqq \rho \leqq 1$, $z = 0$ by the circular annulus $\varepsilon \leqq \rho \leqq 1$, $z = 0$. In the first problem considered we then have to replace the set of equations (1.1.8) by the set

$$\left.\begin{array}{lll} \dfrac{\partial V}{\partial z} = 0, & 0 \leqq \rho < \varepsilon, & 0 \leqq \vartheta < 2\pi; \\ V = f(\rho,\vartheta), & \varepsilon \leqq \rho \leqq 1, & 0 \leqq \vartheta < 2\pi; \\ \dfrac{\partial V}{\partial z} = 0, & \rho > 1, & 0 \leqq \vartheta < 2\pi \end{array}\right\} \tag{1.1.16}$$

on the plane $z = 0$. The problem in which a charged annulus is placed inside a grounded cylinder and the one in which two identical charged annuli are placed with their axes coincident lead to sets of equations which are simple generalizations of the sets (1.1.9–12), (1.1.14–15) respectively.

A further set of problems is obtained by replacing the circular disk by a thin spherical cap. The analogue of the first problem considered above is that of determining the electrostatic field due to a thin spherical cap maintained at a prescribed potential. In this case it is more appropriate to use spherical polar coordinates (r, θ, ϑ) referred to the centre of the sphere as origin and the axis

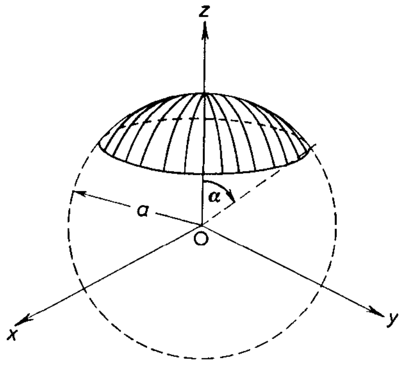

Fig. 5

of the cap (Oz in Fig. 5) as polar axis, and to describe the cap by the equations $r = a$, $0 \leqq \theta \leqq \alpha$. The electrostatic potential V then satisfies Laplace's equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \vartheta^2} = 0 \qquad (1.1.17)$$

and if we write

$$V = \begin{cases} V_1, & 0 \leqq r \leqq a \\ V_2, & r \geqq a \end{cases} \qquad (1.1.18)$$

then the boundary conditions can be put into the form

$$\left.\begin{aligned} V_1 &= V_2 \text{ on } r = a,\ 0 \leqq \theta \leqq \pi; \\ V_1 &= V_2 = f(\theta, \vartheta), \text{ on } r = a,\ 0 \leqq \theta \leqq \alpha,\ 0 \leqq \vartheta \leqq 2\pi; \\ \frac{\partial V_1}{\partial r} &= \frac{\partial V_2}{\partial r} \text{ on } r = a,\ 0 \leqq \theta \leqq \pi; \\ V_1 &= O(r) \text{ as } r \to 0, \\ V_2 &= O(r^{-1}) \text{ as } r \to \infty, \end{aligned}\right\} \qquad (1.1.19)$$

where the function $f(\theta, \vartheta)$ is prescribed for $0 \leqq \theta \leqq \alpha$, $0 \leqq \vartheta < 2\pi$.

In a similar way we can formulate the boundary value problem corresponding to the determination of the electrostatic field due to a charged spherical cap situated inside a grounded cylinder and the problem corresponding to two identical spherical caps placed with their centres and axes coinciding.

Analogous to the circular annulus we could consider the annular spherical cap defined by the equations $r = a$, $\beta \leqq \theta \leqq \alpha$, $0 \leqq \vartheta \leqq 2\pi$ (cf. Fig. 6). The potential function V describing the electrostatic field produced by charging

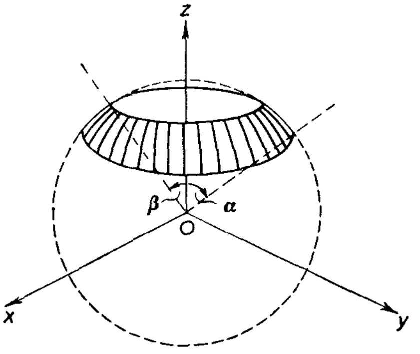

Fig. 6

such a spherical ring to a prescribed potential $f(\theta,\vartheta)$ again must satisfy equation (1.1.17) and if we make the decomposition (1.1.18) it satisfies the set of relations (1.1.19) with the exception that the second and third lines of these conditions are replaced by the conditions

$$\left.\begin{aligned} \frac{\partial V_1}{\partial r} &= \frac{\partial V_2}{\partial r}, & 0 \leqq \theta < \beta,\ & 0 \leqq \vartheta \leqq 2\pi, \\ V_1 &= V_2 = f(\theta,\vartheta), & \beta \leqq \theta \leqq \alpha,\ & 0 \leqq \vartheta < 2\pi, \\ \frac{\partial V_1}{\partial r} &= \frac{\partial V_2}{\partial r}, & \alpha < \theta \leqq \pi,\ & 0 \leqq \vartheta < 2\pi \end{aligned}\right\} \qquad (1.1.20)$$

on the sphere $r = a$.

1.2. Steady-State Diffusion Problems

Mixed boundary value problems arise naturally also in the theory of diffusion —for instance, in the theory of the conduction of heat, or the diffusion of thermal neutrons. For simplicity, we shall restrict our remarks to the case of steady-state conduction of heat. We know from the theory of the conduction of heat that the temperature $\theta(\boldsymbol{r})$ at a point with position vector $\boldsymbol{r}$ in a region R obeys Poisson's equation

$$\kappa\Delta\theta + \Theta = 0 \qquad (1.2.1)$$

where κ is a physical quantity characteristic of the properties of the solid body forming R, and Θ is a source function characterizing the input of energy into the solid from thermal sources situated within it. Now the flux of heat across a small area δS of the surface S of the solid can easily be shown to be

$$\frac{1}{\kappa}\frac{\partial\theta}{\partial n}\delta S,$$

where n denotes distance along the outward drawn normal (cf. Fig. 7). Suppose, for instance, that we have a problem in which we wish to determine the distribution of temperature in a region R, bounded by a simple closed

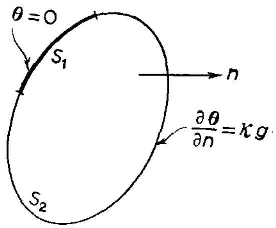

Fig. 7

surface S in the interior of which heat is generated and through the surface of which heat is leaving at a steady rate $g(\boldsymbol{r})$, $\boldsymbol{r} \in S$. Suppose, further, that this temperature field is disturbed by the introduction of a certain distribution of heat sinks on a part S_1 of S such that θ is zero on S_1. We can then readily show that the temperature $\theta(\boldsymbol{r})$ must satisfy the partial differential equation (1.2.1) and the mixed boundary conditions

$$\left.\begin{aligned} \theta(\boldsymbol{r}) &= 0, & \boldsymbol{r} \in S_1, \\ \frac{\partial\theta}{\partial n} &= \kappa g(\boldsymbol{r}), & \boldsymbol{r} \in S_2, \end{aligned}\right\} \tag{1.2.2}$$

where $S = S_1 \cup S_2$.

A problem of this type arose, for instance, in a metallurgical investigation (Karush and Young, 1952); in this instance, R was the half-space $z > 0$, S was the plane $z = 0$ (and a hemisphere of infinite radius), and two cases of S_1 were considered:

(i) S_1 was the infinite strip $|x| \leq a$;
(ii) S_1 was the circle $x^2 + y^2 \leq a^2$.

Another type of problem arises in the case where there are no heat sources within the body. In this connection the function Θ occurring in equation (1.2.1) is identically zero. The temperature field is supposed to be produced by uneven heating of the surface of the body. A typical case is shown in Fig. 8. Over the part S_1 of the surface, the temperature θ is prescribed, while over the remaining

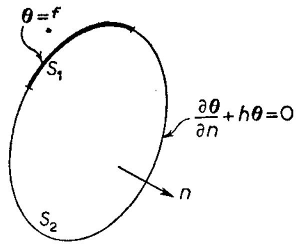

Fig. 8

part of the surface there is radiation from the surface into a medium which is maintained at a fixed temperature (which may be taken to be the zero temperature from which fluctuations are measured). The boundary conditions then become:

$$\left.\begin{aligned} \theta(\boldsymbol{r},t) &= f(\boldsymbol{r}), & \boldsymbol{r}\in S_1, \\ \frac{\partial\theta}{\partial n}+h\theta &= 0, & \boldsymbol{r}\in S_2. \end{aligned}\right\} \qquad (1.2.3)$$

In the first of the equations (1.2.3), the function $f(\boldsymbol{r})$ is assumed to be prescribed, and in the second equation (which is merely an expression of Newton's law of cooling) h is a constant.

Among the simplest boundary value problems in the theory of diffusion are those concerned with the steady flow of heat in cylinders. If $\theta(\rho,\vartheta,z)$ is the deviation of the temperature at the point with cylindrical coordinates (ρ,ϑ,z) from a standard temperature θ_0 then it is well-known that in the steady state θ must be a harmonic function in the region considered, i.e. $\Delta\theta=0$ in the region R under consideration. If we consider the distribution of temperature in the cylinder $\rho\leqq a, 0\leqq z\leqq h$ when the temperature is prescribed over the circle $\rho\leqq 1$ of the flat surface $z=0$ then we have the boundary condition

$$\theta(\rho,\vartheta,0)=f(\rho,\vartheta), \qquad 0\leqq\rho\leqq 1, \qquad 0\leqq\vartheta\leqq 2\pi. \qquad (1.2.4a)$$

If the remaining part of this flat surface is insulated to prevent the flow of heat across it or if there is radiation from that part of the surface into a medium maintained at the standard temperature θ_0 then we have a boundary condition of the form