Lynn Arthur Steen
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Counterexamples in Topology

Second Edition

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Preface

The creative process of mathematics, both historically and individually, may be described as a counterpoint between theorems and examples. Although it would be hazardous to claim that the creation of significant examples is less demanding than the development of theory, we have discovered that focusing on examples is a particularly expeditious means of involving undergraduate mathematics students in actual research. Not only are examples more concrete than theorems—and thus more accessible—but they cut across individual theories and make it both appropriate and necessary for the student to explore the entire literature in journals as well as texts. Indeed, much of the content of this book was first outlined by undergraduate research teams working with the authors at Saint Olaf College during the summers of 1967 and 1968.

In compiling and editing material for this book, both the authors and their undergraduate assistants realized a substantial increment in topological insight as a direct result of chasing through details of each example. We hope our readers will have a similar experience. Each of the 143 examples in this book provides innumerable concrete illustrations of definitions, theorems, and general methods of proof. There is no better way, for instance, to learn what the definition of metacompactness really means than to try to prove that Niemytzki's tangent disc topology is not metacompact.

The search for counterexamples is as lively and creative an activity as can be found in mathematics research. Topology particularly is replete with unreported or unsolved problems (do you know an example of a Hausdorff topological space which is separable and locally compact, but not σ-compact?), and the process of modifying old examples or creating new ones requires a wild and uninhibited geometric imagination. Far from providing all relevant examples, this book provides a context in which to ask new questions and seek new answers. We hope that each reader will share (and not just vicariously) in the excitement of the hunt.

Counterexamples in Topology was originally designed, not as a text, but as a course supplement and reference work for undergraduate and graduate students of general topology, as well as for their teachers. For such use, the reader should scan the book and stop occasionally for a guided tour of the various examples. The authors have used it in this manner as a supplement to a standard textbook and found it to be a valuable aid.

There are, however, two rather different circumstances under which this monograph could most appropriately be used as the exclusive reference in a topology course. An instructor who wishes to develop his own theory in class lecture may well find the succinct exposition which precedes the examples an appropriate minimal source of definitions and structure. On the other hand, *Counterexamples in Topology* may provide sufficiently few proofs to serve as a basis for an inductive, Moore-type topology course. In either case, the book gives the instructor the flexibility to design his own course, and the students a wealth of historically and mathematically significant examples.

A counterexample, in its most restricted sense, is an example which disproves a famous conjecture. We choose to interpret the word more broadly, particularly since all examples of general topology, especially as viewed by beginning students, stand in contrast to the canon of the real line. So in this sense any example which in some respect stands opposite to the reals is truly a *Gegenbeispiel*. Having said that, we should offer some rationale for our inclusions and omissions. In general we opted for examples which were necessary to distinguish definitions, and for famous, well known, or simply unusual examples even if they exhibited no new properties. Of course, what is well known to others may be unknown to us, so we acknowledge with regret the probable omission of certain deserving examples.

In choosing among competing definitions we generally adopted the strategy of making no unnecessary assumptions. With rare exception therefore, we define all properties for all topological spaces, and not just for, for instance, Hausdorff spaces.

Often we give only a brief outline or hint of a proof; this is intentional, but we caution readers against inferring that we believe the result trivial. Rather, in most cases, we believe the result to be a worthwhile exercise which could be done, using the hint, in a reasonable period of time. Some of the more difficult steps are discussed in the Notes at the end of the book.

The examples are ordered very roughly by their appropriateness to the definitions as set forth in the first section. This is a very crude guide whose only reliable consequence is that the numerical order has no correlation with the difficulty of the example. To aid an instructor in recommending examples for study, we submit the following informal classification by sophistication:

Elementary:	1–25, 27–28, 30–34, 38, 40–47, 49–50, 52–59, 62–64, 73–74, 81, 86–89, 97, 104, 109, 115–123, 132–135, 137, 139–140.
Intermediate:	26, 29, 35–37, 39, 48, 51, 65–72, 75–80, 82–85, 90–91, 93–96, 98–102, 105–108, 113–114, 124, 126–127, 130, 136, 138, 141.
Advanced:	60–61, 92, 103, 110–112, 125, 128–129, 131, 142, 143.

The discussion of each example is geared to its general level: what is proved in detail in an elementary example may be assumed without comment in a more advanced example.

In many ways the most useful part of this book for reference may be the appendices. We have gathered there in tabular form a composite picture of the most significant counterexamples, so a person who is searching for Hausdorff nonregular spaces can easily discover a few. Notes are provided which in addition to serving as a guide to the Bibliography, provide added detail for many results assumed in the first two sections. A collection of problems related to the examples should prove most helpful if the book is used as a text. Many of the problems ask for justification of entries in the various tables where these entries are not explicitly discussed in the example. Many easy problems of the form "justify the assertion that . . ." have not been listed, since these can readily be invented by the instructor according to his own taste.

In most instances, the index includes only the initial (or defining) use of a term. For obvious reasons, no attempt has been made to include in the index all occurrences of a property throughout the book. But the General Reference Chart (pp. 170–179) provides a complete cross-tabulation of examples with properties and should facilitate the quick location of examples of any specific type. The chart was prepared by an IBM 1130 using a program which enables the computer to derive, from the theorems discussed in Part I, the properties for each example which follow logically from those discussed in Part II.

Examples are numbered consecutively and referred to by their numbers in all charts. In those few cases where a minor but inelegant modification of an example is needed to produce the desired concatenation of properties, we use a decimal to indicate a particular point within an example: 23.17 means the 17th point in Example 23.

The research for this book was begun in the summer of 1967 by an undergraduate research group working with the authors under a grant from the National Science Foundation. This work was continued by the authors with support from a grant by the Research Corporation, and again in the summer of 1968 with the assistance of an N.S.F. sponsored undergraduate research group. The students who participated in the undergraduate research groups were John Feroe, Gary Gruenhage, Thomas Leffler, Mary Malcolm, Susan Martens, Linda Ness, Neil Omvedt, Karen Sjoquist, and Gail Tverberg. We acknowledge that theirs was a twofold contribution: not only did they explore and develop many examples, but they proved by their own example the efficacy of examples for the undergraduate study of topology.

Finally, we thank Rebecca Langholz who with precision, forbearance, and unfailing good humor typed in two years three complete preliminary editions of this manuscript.

Northfield, Minn.
January 1970

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Preface to the Second Edition

In the eight years since the original edition of *Counterexamples* appeared, many readers have written pointing out errors, filling in gaps in the reference charts, and supplying many answers to the rhetorical question in our preface. In these same eight years research in topology produced many new results on the frontier of metrization theory, set theory, topology and logic.

This *Second Edition* contains corrections to errors in the first edition, reports of recent developments in certain examples with current references, and, most importantly, a revised version of the first author's paper "Conjectures and counterexamples in metrization theory" which appeared in the *American Mathematical Monthly* (Vol. 79, 1972, pp. 113–132). This paper appears as Part III of this *Second Edition* by permission of the Mathematical Association of America.

We would like to thank all who have taken the time to write with corrections and addenda, and especially Eric van Douwen for his extensive notes on the original edition which helped us fill in gaps and correct errors. The interest of such readers and of our new publisher Springer-Verlag has made this second edition possible.

Northfield, Minn.
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PART I
Basic Definitions

SECTION 1
General Introduction

A **topological space** is a pair (X,τ) consisting of a set X and a collection τ of subsets of X, called **open sets**, satisfying the following axioms:

O_1: The union of open sets is an open set.
O_2: The finite intersection of open sets is an open set.
O_3: X and the empty set $\emptyset$ are open sets.

The collection τ is called a **topology** for X. The topological space (X,τ) is sometimes referred to as the **space** X when it is clear which topology X carries.

If τ_1 and τ_2 are topologies for a set X, τ_1 is said to be **coarser** (or **weaker** or **smaller**) than τ_2 if every open set of τ_1 is an open set of τ_2. τ_2 is then said to be **finer** (or **stronger** or **larger**) than τ_1, and the relationship is expressed as $\tau_1 \leq \tau_2$. Of course, as sets of sets, $\tau_1 \subseteq \tau_2$. On a set X, the coarsest topology is the indiscrete topology (Example 4), and the finest topology is the discrete topology (Example 1). The ordering $\leq$ is only a partial ordering, since two topologies may not be **comparable** (Example 8.8).

In a topological space (X,τ), we define a subset of X to be **closed** if its complement is an open set of X, that is, if its complement is an element of τ. The De Morgan laws imply that closed sets, being complements of open sets, have the following properties:

C_1: The intersection of closed sets is a closed set.
C_2: The finite union of closed sets is a closed set.
C_3: X and the empty set $\emptyset$ are both closed.

It is possible that a subset be both open and closed (Example 1), or that a subset be neither open nor closed (Examples 4 and 28).

An F_σ**-set** is a set which can be written as the union of a countable collection of closed sets; a **G_δ-set** is a set which can be written as the intersection of a countable collection of open sets. The complement of every F_σ-set is a G_δ-set and conversely. Since a single set is, trivially, a countable collection of sets, closed sets are F_σ-sets, but not conversely (Example 19). Furthermore, closed sets need not be G_δ-sets (Example 19). By complementation analogous statements hold concerning open sets.

Closely related to the concept of an open set is that of a **neighborhood**. In a space (X,τ), a neighborhood N_A of a set A, where A may be a set consisting of a single point, is any subset of X which contains an open set containing A. (Some authors require that N_A itself be open; we call such sets **open neighborhoods**.) A set which is a neighborhood of each of its points is open since it can be expressed as the union of open sets containing each of its points.

Any collection $\mathcal{S}$ of subsets of X may be used as a **subbasis** (or **subbase**) to generate a topology for X. This is done by taking as open sets of τ all sets which can be formed by the union of finite intersections of sets in $\mathcal{S}$, together with $\varnothing$ and X. If the union of subsets in a subbasis $\mathcal{S}$ is the set X and if each point contained in the intersection of two subbasis elements is also contained in a subbasis element contained in the intersection, $\mathcal{S}$ is called a **basis** (or **base**) for τ. In this case, τ is the collection of all sets which can be written as a union of elements of $\mathcal{S}$. Finite intersections need not be taken first, since each finite intersection is already a union of elements of $\mathcal{S}$. If two bases (or subbases) generate the same topology, they are said to be **equivalent** (Example 28). A **local basis** at the point $x \in X$ is a collection of open neighborhoods of x with the property that every open set containing x contains some set in the collection.

Given a topological space (X,τ), a topology τ_Y can be defined for any subset Y of X by taking as open sets in τ_Y every set which is the intersection of Y and an open set in τ. The pair (Y,τ_Y) is called a **subspace** of (X,τ), and τ_Y is called the **induced** (or **relative**, or **subspace**) **topology** for Y. A set $U \subset Y$ is said to have a particular property **relative to Y** (such as open relative to Y) if U has the property in the subspace (Y,τ_Y). A set Y is said to have a property which has been defined only for topological spaces if it has the property when considered as a subspace. If for a particular property, every subspace has the property whenever a space does, the property is said to be **hereditary**. If every closed subset when considered as a subspace has a property whenever the space has that property, that property is said to be **weakly hereditary**.

An important example of a weakly hereditary property is compactness. A space X is said to be **compact** if from every **open cover**, that is, a collection of open sets whose union contains X, one can select a finite subcollection whose union also contains X. Every closed subset Y of a compact space is compact, since if $\{O_\alpha\}$ is an open cover for Y, $\{O_\alpha\} \cup (X - Y)$ is an open cover for X. From $\{O_\alpha\} \cup (X - Y)$, one can choose a finite subcollection covering X, and from this one can choose an appropriate cover for Y containing only elements of $\{O_\alpha\}$ simply by omitting $X - Y$. A compact subset of a compact space need not be closed (Examples 4, 18).

Limit Points

A point p is a **limit point** of a set A if every open set containing p contains at least one point of A distinct from p. (If the point of A is not required to be distinct from p, p is called an **adherent point**.) Particular kinds of limit points are **ω-accumulation points**, for which every open set containing p must contain infinitely many points of A; and **condensation points**, for which every open set containing p must contain uncountably many points of A. Examples 8 and 32 distinguish these definitions.

The concept of limit point may also be defined for sequences of not necessarily distinct points. A point p is said to be a **limit point of a sequence** $\{x_n\}$, $n = 1, 2, 3, \ldots$, if every open set containing p contains all but finitely many terms of the sequence. The sequence is then said to **converge** to the point p. A weaker condition on p is that every open set containing p contains infinitely many terms of the sequence. In this case, p is called an **accumulation point of the sequence**. It is possible that a sequence has uncountably many limit points (Example 4), both a limit point and an accumulation point that is not a limit point (Example 53), or a single accumulation point that is not a limit point (Example 28).

Since a sequence may be thought of as a special type of ordered set, each sequence has associated with it, in a natural way, the set consisting of its elements. On the other hand, every countably infinite set has associated with it many sequences whose terms are points of the set. There is little relation between the limit points of a sequence and the limit points of its associated set. A point may be a limit point of a sequence, but only an adherent point of the associated set (Example 1). If the points of the sequence are distinct, any accumulation point (and therefore any limit point) of the sequence is an ω-accumulation point of the associated set. Likewise, any ω-accumulation point of a countably infinite set is also an accumulation point (but not necessarily a limit point) of any sequence corresponding to the set. Not too surprisingly, a point may be a limit point of a countably infinite set, but a corresponding sequence may have no limit or accumulation point (Example 8).

If A is a subset of a topological space X, the **derived set** of the set A is the collection of all limit points of A. Generally this includes some points