GROUP STRUCTURE OF GAUGE THEORIES

L. O'RAIFEARTAIGH

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L. O'RAIFEARTAIGH
Dublin Institute for Advanced Studies

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Preface

It has been known for many years that the gravitational and electromagnetic interactions of matter can be formulated as gauge theories – based on the Lorentz group $SO(3,1)$ and the compact 'internal' phase group $U(1)$, respectively. But over the past two decades it has gradually come to be accepted that the remaining two (known) fundamental interactions of matter, namely the strong and weak nuclear interactions, are also gauge interactions, a property that had been hidden by confinement for the strong interactions and by spontaneous symmetry breaking for the weak ones. To be more precise, it has now been established beyond reasonable doubt that the weak nuclear interactions combine with electromagnetism to form a gauge interaction based on the compact internal non-abelian group $U(2)$, and, although the evidence is less direct, it is accepted that the strong interactions are gauge interactions based on the compact simple internal (colour) group $SU(3)$. By combining these results one sees that the (known) non-gravitational interactions may be described by a gauge theory based on a compact internal group with Lie algebra $SU(3) \times SU(2) \times U(1)$. (The global group is actually $S(U(3) \times U(2))$ because of certain discrete correlations in the particle classification, chapter 9.)

If the $S(U(3) \times U(2))$ theory of the non-gravitational interactions is correct, it represents an immense advance because gauge theories, by their nature, determine the form of the interactions, leaving only a finite number of constants as free parameters, and thus in principle at least, the form of all the fundamental interactions is now known. Furthermore, since gauge theories have a geometrical interpretation in terms of fibre bundles, it means that even the non-gravitational interactions have a geometrical significance and are thus brought a step nearer to gravitation.

On the other hand the common gauge structure of all the interactions does not mean that the interactions are fully unified, because the gravitational interaction has special properties not shared by the others (the existence of the metric and the equivalence principle, for example) and the three other interactions remain separate in the sense that the $SU(3) \times SU(2) \times U(1)$ algebra consists of three irreducible pieces, with a separate coupling constant for each piece. For this reason it has been suggested that the gauge group $S(U(3) \times U(2))$ is actually only a subgroup of a larger, simple, compact gauge group G, which has only one coupling constant and which truly unifies the three non-gravitational interactions. Theories based on such groups G are called grand unified theories (GUTs) and have been extensively studied in recent years. Although the most spectacular prediction of GUTs, namely proton decay, has not yet been (and may even never be) observed, there is a certain amount of indirect evidence for GUTs (chapter 10), notably from the particle classification, from renormalization group considerations and from cosmology.

Compact gauge theories are, in principle, generalizations of electromagnetism from $U(1)$ to non-abelian groups, but the generalization is not trivial for two reasons. First, the intrinsic group structure (Lie algebras, representations, invariants, etc.) is much more complicated than in the abelian case. Second, spontaneous symmetry breaking, which enters only in the special case of superconductivity for electromagnetism, plays a central role for the non-abelian theories.

The aim of the present monograph is to provide a review of the group structure both of the non-abelian gauge theories themselves and of their spontaneous symmetry breaking. The presentation is pitched at about the graduate student level and, so as not to overlap with the many excellent treatments of other aspects of gauge theories (renormalization, phenomenology, confinement, topology, etc.), it concentrates on two aspects. These are the group theoretical background, particularly the global group theory (Part I) and the algebraic structure of the gauge interactions and of their symmetry breakdown patterns (Part II). The spontaneous symmetry breaking is treated in some detail (at the classical level) because many results in this area have not previously been available in book form. It should be stated, however, that the investigation of symmetry breaking patterns is still at an early stage of development and so the results presented should be regarded as pioneering ones.

The general plan of the monograph may be seen from the list of contents, but a few remarks may be in order. In chapters 1–5, where the group-theoretical background is given, some of the more technical equipment (tables of branching rules and Clebsch–Gordon coefficients for example) has been omitted because it is available elsewhere and space did not permit a reasonable resumé. In the chapters on spontaneous symmetry breaking (8, 11, 12) it is assumed, for definiteness, that the symmetry breakdown is caused by a local scalar potential, but it is fairly evident that because of the group-theoretical nature of the results most of them would survive in a much broader context, e.g. if the scalar field were composite. Indeed this is one of the justifications for the group-theoretical approach. With regard to the references, and the suggestions for further reading, the literature on both Lie groups and gauge theory is so vast there was no hope of providing a comprehensive bibliography, and accordingly these sections have been limited to those references which are strictly relevant, to recent reviews (many of which, notably Langacker (1981), contain extensive lists of references) and to books.

Finally, I should like to take this opportunity to thank Professors Nikolas Kuiper (director) and Louis Michel for their kind hospitality at the Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, for most of the academic year 1983–4, when much of the monograph was written. I should also thank Louis Michel, whose influence pervades not only the book but the whole literature on symmetry and symmetry breaking, for many invaluable discussions and comments.

L. O'RAIFEARTAIGH *Dublin, 1985*

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Part I
Group structure

1
Global properties of groups and Lie groups

1.1 Groups

Part I of this monograph is concerned mainly with the theory of Lie groups, which is the background group theory necessary for gauge interactions. But since some of the relevant properties of Lie groups are common to many topological groups, and even to groups in general, it is convenient to begin with a brief discussion of general and topological groups.

First, a group G is defined as a set of elements $\{g\}$ (not necessarily finite or even countable) for which there is a multiplication law $G \times G \to G$ ($g_1 g_2 \in G$ for $g_1, g_2 \in G$) with the three properties:

(i) Associativity, $g_1(g_2 g_3) = (g_1 g_2) g_3$;
(ii) Existence of an identity element e, such that, for each $g \in G$, $eg = ge = g$;
(iii) Existence of an inverse element g^{-1} such that, for each $g \in G$, $g^{-1}g = gg^{-1} = e$.

The identity e and the inverse g^{-1} are easily seen to be unique. In physics and geometry, groups usually occur as transformations which leave some quantity (or set of quantities) invariant, simply because the product of any two such transformations also leaves the quantity invariant. In particular, groups of transformations that leave the Hamiltonian or Lagrangian invariant are called symmetry groups.

An important concept which arises for group transformations is that of group *orbits*, which may be defined as follows: let a group G of transformations act on a set of elements $\{s\}$. Then the subset of all elements that can be obtained from any given element s_0 by the action of G is called the *orbit* of s_0 with respect to G, or the G-orbit of s_0. For example, the orbits of the rotation group in Euclidean 3-space are the surfaces of the spheres of radius r for each $0 \leqslant r < \infty$. The group itself is an orbit with respect to left or right multiplication, and group invariants are trivial (one-element) orbits. The action of the group on an orbit is transitive (i.e. any element on an orbit can be obtained from any other one by a group transformation) and membership of an orbit is a class relation ($s \in O(s_0) \Rightarrow s_0 \in O(s)$ and $s \in O(s'), s' \in O(s'') \Rightarrow s \in O(s'')$). Thus any set on which a group acts can be partitioned into distinct group orbits.

The set of elements z that commute with all other group elements ($zg = gz$ all $g \in G$) is called the *centre* Z of G, and it includes at least the identity e. If all elements commute ($Z = G$) the group is called *abelian*.

A *subgroup* H of a group G is a subset of $\{h\}$ of elements of G which closes with respect to the multiplication already defined by G ($hk \in H$ for $h, k \in H$) and which contains the inverse of each of its elements h, and the identity e. Subgroups usually arise by making a natural restriction of the original group, e.g. restricting the group of rotations in three dimensions to rotations about one axis, or to discrete rotations.

One of the most important group operations is *conjugation*. The conjugation of one element g by another h is the transformation $g \to hgh^{-1}$. The conjugation of the group G by a single element h is the transformation $g \to hgh^{-1}$ for all $g \in G$ and it preserves the group multiplication since $gg' \to h(gg')h^{-1} = (hgh^{-1})(hg'h^{-1})$. The set of elements obtained by the conjugation of a fixed element h by the whole of G, ghg^{-1} for all $g \in G$, is called the conjugation *class* of h, and, since the class is a group orbit, a group can be partitioned into distinct conjugacy classes. For example, for the permutation group of three objects (in standard notation), the conjugacy classes are $\{(1), (2), (3)\}$ $\{(12), (13), (23)\}$ and $\{(123), (132)\}$. Since $geg^{-1} = e$, the identity element forms a separate conjugacy class, and more generally, a given element forms a separate conjugacy class if, and only if, it lies in the centre Z. In physical or geometrical situations the elements of a conjugacy class usually have some obvious physical or geometrical characteristic in common. For example, for the rotation group all rotations of the same magnitude (but in different directions) are in the same conjugacy class.

An invariant subgroup H of G is one which is invariant with respect to conjugation with G, i.e. $ghg^{-1} \in H$ for all $h \in H$, $g \in G$. For example, the translation subgroups of the space–time groups are invariant subgroups because they are transformed into themselves by rotations.

To each subgroup H of a group G is associated its right (left) cosets, where the cosets are the G-orbits of H with respect to right (left) multiplication. In other words, g_1 and g_2 are in the same right (left) coset if, and only if, there exists an element h of H such that $g_1 = g_2 h$ (hg_2). From the properties of orbits it follows that the group may be partitioned into distinct cosets, and because $g_1 = hg_2 \Leftrightarrow h = g_1 g_2^{-1}$ and $g_1 g_2^{-1} = e \Leftrightarrow g_1 = g_2$ the dimension of each coset, including H itself, is the same. For finite groups the equidimensionality of the cosets implies that dim G/dim H = integer (= number of cosets) and thus gives a limitation on the number of subgroups. In geometrical or physical situations the cosets are often parametrized in a direct geometrical or physical manner. For example, for fixed mass, the cosets of the rotation subgroup $SO(3)$ of the Lorentz group $SO(3, 1)$ are often parametrized by the 3-momentum $\mathbf{p}$.

In general the left and right cosets of a subgroup H are not identical. If they are, and if h and g are any elements of H and G respectively, then $gh = h'g$ for some $h' \in H$ and thus $ghg^{-1} = h'$, which is just the condition that H be an invariant subgroup. Conversely, if $ghg^{-1} = h' \in H$ for all $g \in G$, $h \in H$, then $gh = h'g$. Thus the necessary and sufficient condition for the left and right cosets of a subgroup to be the same is that the subgroup be invariant.

When H is an invariant subgroup the cosets c themselves form a group, called the quotient group $Q = G/H$. For the quotient group the identity element is H itself and the multiplication is that induced by G, i.e. $cc' = c''$ where $g \in c$, $g' \in c'$ and c'' contains gg'. Note that invariance of H is necessary for the consistency of this scheme. The quotient group is not a subgroup of G, nor is it, in general, isomorphic to a subgroup of G. An interesting example from physics is the Heisenberg group of real upper triangular 3×3 matrices:

$$H(\alpha, \beta, m) = \begin{pmatrix} 1 & \alpha & m \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.1}$$

The subgroup $\alpha = \beta = 0$ is an invariant (even central) subgroup, and the quotient group is the two-dimensional translation group

$$T(\alpha, \beta) \approx \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

But the quotient group is not a subgroup of $H(\alpha, \beta, m)$ and both the invariant subgroup and the quotient group are abelian although $H(\alpha, \beta, m)$ itself is not.

A mapping of a group onto itself ($g \to g'(g) \in G$) which preserves the multiplication law is called an *automorphism* of the group. For example, for the group of unitary unimodular complex $n \times n$ matrices $SU(n)$, $n \geqslant 2$, complex conjugation is an automorphism, but hermitian conjugation is not because it reverses the order of multiplication. It has been seen already that the conjugation of a group by a fixed element preserves the multiplication law. Thus conjugation with any element is an automorphism, and any automorphism that can be implemented by a conjugation is called an *inner automorphism*. Thus complex conjugation $M \to M^*$ is inner for $SU(2)$ since $M^* = CMC^{-1}$ where $M \in SU(2)$ and $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU(2)$, but is not inner for $SU(n)$, $n \geqslant 3$ because for $n \geqslant 3$ there are elements such that $\operatorname{tr} M \neq \operatorname{tr} M^*$ and for such elements M and M^* cannot be conjugate. The sets of all automorphisms (Aut(G)) and of all inner automorphisms (Int(G)) of a group G are themselves groups.

A map from a group G onto (or indeed into) another group G' such that the multiplication law is preserved is called a homomorphism of G, and G' is called the homomorphic image. The set of elements of G which map onto the identity element e' of G' is called the *kernel* of the homomorphism. For example, the rotation group $SO(3)$ in three dimensions is a homomorphic image of the Euclidean group $E(3)$ (rotations and translations) with the translation subgroup of $E(3)$ as the kernel. It is not difficult to see that the kernel of a homomorphism $G \to G'$ must be an invariant subgroup of G. If the homomorphism between G and G' is one–one the two groups are said to be *isomorphic*.

A *direct product* $G = A \times B$ of two groups A and B is a group whose elements are $g = (a, b)$ for all $a \in A$, $b \in B$ and whose multiplication law is $gg' = (aa', bb')$. A somewhat more subtle structure is that of a *semi-direct* product. A fairly familiar occurrence of this structure is in the case of the Euclidean group $E(3)$ and its various crystallographic (finite) subgroups, all of which are semi-direct products of rotation groups R and translation groups T. That means that each element g of the group can be expressed as $g = (r, t)$ where r is a rotation and t is a translation and the multiplication law is $gg' = (r, t)(r', t') = (rr', t + rt')$ where rt' denotes the translation t' rotated by r. Note that the rotations affect the translations but not vice versa. The generalization of this law for a semi-direct product $A \wedge B$ of any two groups A and B is $g = (a, b)$ where a and b are elements of A and B and $gg' = (a, b)\ (a', b') = (aa', bh(a)b')$ where the transformations $b \to h(a)b$ are automorphisms of B, i.e. $(h(a)b)\ (h(a)b') = h(a)bb'$ and are homomorphisms (representations) of A, i.e. $h(aa')b = h(a)\ (h(a')b)$. This multiplication law satisfies the associativity condition $g(g'g'') = (g'g)g''$ (exercise 1.1), a result which is evident in the special case that the automorphisms are inner, i.e. each $a \in A$ has an image $b(a)$ in B and $h(a)b = b(a)bb^{-1}(a)$. The subgroup $\tilde{B} = (1, B)$ of a semi-direct product $A \wedge B$ is evidently an invariant subgroup, and the quotient group $G/\tilde{B}$ is isomorphic to A. Conversley a group G with an invariant subgroup B is isomorphic to a semi-direct product $(G/B) \wedge B$ if the quotient group G/B is isomorphic to a subgroup of G. An interesting example of a semi-direct product for compact Lie groups is the semi-direct product $W \wedge H$ of the Weyl group W and the Cartan group H (section 2.3). This group is the normalizer of the Cartan algebra, i.e. it is the maximal subgroup of the compact Lie group which conjugates the Cartan algebra into intself.

1.2 Topological groups

When the elements of a group are not denumerable it is often desirable to have a notion of continuity, and so a topology is introduced. A topological group is defined to be a group with any topology in which the multiplication and inversion are continuous. That is to say, if g and h are in the neighbourhood of g' and h' respectively, in the given topology, then gh and g^{-1} are in the neighbourhood of $g'h'$ and $(g')^{-1}$. In practice, a topology is often suggested by the context in which the group is considered. Thus, for example, a natural topology for a group of real matrices is the usual topology of the real line for the matrix elements.

The most important topological concept that will be needed is that of *compactness*. It will be recalled that a topological space is compact if every covering (set of open sets containing every point) has a finite subcovering. In particular, for a space whose topology is induced by a metric, compactness is equivalent to the statement that the space is closed and bounded with respect to that metric. A much weaker, but very important form of compactness is *local* compactness, and a topological space is said to be locally compact if every neighbourhood of a point contains a compact subneighbourhood. Lie groups, which are locally Euclidean, are locally compact but not necessarily compact.

One of the first uses of a topology is to define the connectivity structure of a group. Two elements g and h are said to be connected, if for a real parameter $0 \leqslant t \leqslant 1$ there exists in G a continuous path $g(t)$ with $g(0) = g$ and $g(1) = h$. The connectedness of elements is a class relation and hence a topological group may be partitioned into disjoint self-connected components. The component G_0 containing the identity element e is called the identity component. For example, the group $O(n)$ of real orthogonal $n \times n$ matrices, $O^t(n)\, O(n) = 1$, where t denotes transpose, consists of two disconnected components $\det O(n) = 1$ and $\det O(n) = -1$, the first, called $SO(n)$, being the identity component.

In the $O(n)$ example one sees that the identity component is an invariant subgroup ($\det N = 1$ implies $\det NK = 1$ for $\det K = 1$ and $\det MNM^{-1} = 1$ for $\det M = \pm 1$), and it turns out that this result is completely general: *the identity component of any topological group is an* *invariant subgroup*. The proof is quite straightforward. First, the connected component forms a subgroup because, if g and h are connected to e (by $g(t)$ and $h(t)$), then gh is connected to e (by $g(t)\,h(t)$). Second, the subgroup is invariant because, for g connected to e and any $k \in G$, kgk^{-1} is connected to e (by $kg(t)\,k^{-1}$).

The other components are the cosets of G_0 because for g, h in the same component (connected to each other by $k(t)$ say) $g^{-1}h$ is in the connected component (connected to e by $g^{-1}k(t)$). Thus the components form a discrete quotient group $D = G/G_0$. In many cases the group G is actually a direct or semi-direct product of the form $D \times G_0$ or $D \wedge G_0$. For example for the orthogonal groups, $O(2n+1)$ is a direct product of the form $Z_2 \times SO(2n+1)$ where Z_2 is the two-element group ± 1, and $O(2n)$ is a semi-direct product of the form $Z_2 \wedge SO(2n)$ where Z_2 is the two-element group $\mathrm{diag}(1, 1, \ldots, 1, 1, \pm 1)$. However, G is not always a direct or semi-direct product, even for compact groups. For compact Lie groups a complete analysis of the disconnected structure has been given by de Siebentahl (1956).

A connected group is said to be *simply connected* if each closed continuous curve $g(t)$ $(0 \leqslant t \leqslant 1,\ g(0) = g(1))$ in it may be continuously deformed to zero (by a family of such curves $g(t, s), 0 \leqslant s \leqslant 1$ $g(t, 0) = g(t)$, $g(t, 1) = e$). (For those familiar with homotopy theory (Nash and Sen, 1983) simple connectivity $\Leftrightarrow \pi_1(G) = 0$.) For example the $1+1$ dimensional Lorentz group e^x, $-\infty < x < \infty$ is simply connected, but the rotation group $e^{i\phi}$, $0 \leqslant \phi \leqslant 2\pi$ is not. For Lie groups, simple connectivity will be discussed in more detail in section 1.4. Another important use of a topology is to construct a measure $\mu(g)$ on the group. A measure is desirable because it allows the concept of summing over the elements for finite groups to be generalized to integration for continuous groups. For this reason the measure is required to be invariant with respect to group multiplication (either left or right). That is, it is required to satisfy the relation

$$\int d\mu(g) f(gh) = \int d\mu(g) f(g) \tag{1.2}$$

(and similarly for left multiplication) for every continuous function $f(g)$ of compact support. A sufficient condition for the existence of such a measure is that the group be locally compact (Weil, 1953), in which case the measure is called Haar measure and is unique up to a constant. For abelian groups, the left and right Haar measures are the same, and for the Euclidean translation groups it is just the Lebesgue measure. Similarly, for the Heisenberg group (1.1) both measures are just the Euclidean measure $d\alpha\, d\beta\, dm$, and for the general real linear group $GL(n, r)$ both measures are

$(\det M)^{-1} \prod \mathrm{d}m_{ij}$, where m_{ij} are the matrix elements of M. However, for the real triangular matrix group $\begin{pmatrix} e^a & x \\ 0 & e^b \end{pmatrix}$, the left and right measures are different, being $\exp(-a)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}x$ and $\exp(-b)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}x$ respectively. Since Lie groups are locally compact they always have invariant measures, and an explicit construction will be given in section 2.2.

In the non-abelian case a simple criterion for the left and right Haar measures to be the same is the following: let $\mu(g)$ be a left-invariant measure. Then, for any fixed element k, $\mu(gk)$ is also a left-invariant measure, and hence, by the uniqueness, $\mu(gk) = \lambda(k)\mu(g)$ where $\lambda(k)$ is a positive factor, called the modular factor. Then

$$\int \mathrm{d}\mu(g) f(gk^{-1}) = \int \mathrm{d}\mu(gk) f(g) = \lambda(k) \int \mathrm{d}\mu(g) f(g), \tag{1.3}$$

and so $\mu(g)$ is right-invariant if, and only if, the modular factor is a constant.

Compact groups have the special property that the continuous functions of compact support include $\theta(g) \equiv 1$. Using θ for f in (1.3) one sees at once that $\lambda(k) = 1$ and thus for compact groups the left and right measures are the same. Furthermore, since

$$\int_G \mathrm{d}\mu(g) = \int \mathrm{d}\mu(g)\,\theta(g) < \infty, \tag{1.4}$$

the total measure for compact groups is finite. This is probably the most important property of compact groups and one of its consequences is that the representation theory of compact groups is similar to that of finite groups (chapter 5).

It is often useful to have a left- (right)-invariant *metric* $\rho(g, h) = \rho(kg, kh)$ and a necessary and sufficient condition for this is that the topology have a countable basis. It is necessary because any metrically induced topology has such a basis (e.g. the spheres $\rho(g, g) <$ rationals) and is sufficient because of a theorem due to Birkhoff and Takhutani (see Barut and Raczka 1977). Since Lie groups are locally Euclidean (section 1.3) they satisfy this condition, and the metric will be constructed explicitly in section 2.2.

For compact groups, in particular compact Lie groups, there exist metrics which are both left- and right-invariant. In fact, for compact groups any measurable metric can be converted into a left- or right-invariant one by averaging with the Haar measure. Thus, in particular, if $\rho(f, g)$ is a left-invariant metric,

$$\Delta(f, g) = \int \mathrm{d}\mu(h)\,\rho(fh, gh), \tag{1.5}$$

is again a metric, and is both left- and right-invariant.

Finally it should be mentioned that by subgroups of topological, in particular Lie, groups will be meant closed subgroups, where closed means closed in the topology of the original group. For example, the subgroups obtained by restricting the continuous parameters of Lie groups to the rationals are excluded.

1.3 Lie groups: global considerations

After the above digression into general and topological groups let us consider Lie groups. These are groups for which the topology is locally Euclidean. That is, in the neighbourhood of any point the group may be parametrized by a *finite* number of *continuous* (real) variables. Thus $g = g(a_1, \dots a_r)$, where the a_k, $k = 1, \dots, r$ are continuous and, by convention, $e = g(0, 0, \dots, 0)$. In general the 'rigid' groups of theoretical physics, e.g. the rotation groups, Lorentz group, groups of unitary transformations on finite-dimensional spaces etc. are Lie groups, while the more 'flexible' groups such as the group of all coordinate transformations in general relativity, local gauge groups, and the group of all canonical transformations in classical point mechanics are not Lie groups because the number of parameters is not finite. The archetypal Lie group is an $n \times n$ matrix group, with continuous elements, and indeed it can be shown that every Lie group is isomorphic to a matrix group (at least in the neighbourhood of the identity). Thus it is useful to keep the continuous matrix groups in mind as concrete realizations of Lie groups. Some important examples of continuous matrix groups which are Lie groups are:

(1) $GL(n, c/r)$ = group of all complex/real non-singular $n \times n$ matrices M.
(2) $SL(n, c/r)$ = group of all complex/real unimodular $n \times n$ matrices M ($\det M = 1$).
(3) $D(n, c/r/u)$ = group of all complex/real/unitary diagonal non-singular $n \times n$ matrices.
(4) $T(n, c/r)$ = group of all complex/real upper-triangular non-singular $n \times n$ matrices.
(5) $T_0(n, c/r)$ = group of all complex/real upper-triangular unit-diagonal $n \times n$ matrices.
(6) $U(n)$ = group of all complex unitary $n \times n$ matrices.
(7) $SU(n)$ = group of all complex unitary unimodular $n \times n$ matrices.
(8) $O(n, c/r)$ = group of all complex/real orthogonal $n \times n$ matrices.
(9) $SO(n, c/r)$ = group of all unimodular complex/real orthogonal $n \times n$ matrices.