SCHAUM'S OUTLINE OF

THEORY AND PROBLEMS OF BOOLEAN ALGEBRA and SWITCHING CIRCUITS

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BY

ELLIOTT MENDELSON, Ph.D.

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Preface

This book is devoted to two separate, but related, topics: (1) the synthesis and simplification of switching and logic circuits, and (2) the theory of Boolean algebras.

Those people whose primary interest is in switching and logic circuits can read Chapter 4 immediately after a quick perusal of Chapter 1. We have confined our treatment of switching and logic circuits to combinational circuits, i.e. circuits in which the outputs at a given time depend only on the present value of the inputs and not upon the previous values of the inputs. The extensive theory of sequential circuits, in which the outputs depend also upon the history of the inputs, may be pursued by the reader in *Introduction to Switching Theory and Logical Design* by F. J. Hill and G. R. Peterson (reference 34, page 202), *Introduction to Switching and Automata Theory* by M. A. Harrison (ref. 33, page 202), and other textbooks on switching theory.

The treatment of Boolean algebras is somewhat deeper than in most elementary texts. It can serve as an introduction to graduate-level books such as *Boolean Algebras* by R. Sikorski (ref. 148, page 207) and *Lectures on Boolean Algebras* by P. R. Halmos (ref. 116, page 207).

There is no prerequisite for the reading of this book. Each chapter begins with clear statements of pertinent definitions, principles and theorems together with illustrative and other descriptive material. This is followed by graded sets of solved and supplementary problems. The solved problems serve to illustrate and amplify the theory, bring into sharp focus those fine points without which the student continually feels himself on unsafe ground, and provide the repetition of basic principles so vital to effective learning. A few problems which involve modern algebra or point-set topology are clearly labeled. The supplementary problems serve as a complete review of the material of each chapter. Difficult problems are identified by a superscript ^{D} following the problem number.

The extensive bibliography at the end of the book is divided into two parts, the first on Switching Circuits, Logic Circuits and Minimization, and the second on Boolean Algebras and Related Topics. It was designed for browsing. We have listed many articles and books not explicitly referred to in the body of the text in order to give the reader the opportunity to delve further into the literature on his own.

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ELLIOTT MENDELSON

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Chapter 1

The Algebra of Logic

1.1 TRUTH-FUNCTIONAL OPERATIONS

There are many ways of operating on propositions to form new propositions. We shall limit ourselves to those operations on propositions which are most relevant to mathematics and science, namely, to truth-functional operations. An operation is said to be *truth-functional* if the truth value (truth or falsity) of the resulting proposition is determined by the truth values of the propositions from which it is constructed. The investigation of truth-functional operations is called the *propositional calculus,* or, in old-fashioned terminology, the *algebra of logic,* although its subject matter forms only a small and atypically simple branch of modern mathematical logic.

Negation

Negation is the simplest common example of a truth-functional operation. If **A** is a proposition, then its denial, not-**A**, is true when **A** is false and false when **A** is true. We shall use a special sign ¬ to stand for negation. Thus, ¬**A** is the proposition which asserts the denial of **A**. The relation between the truth values of ¬**A** and **A** can be made explicit by a diagram called a truth table.

A	¬A
T	F
F	T

Fig. 1-1

In this truth table, the column under **A** gives the two possible truth values T (truth) and F (falsity) of **A**. Each entry in the column under ¬**A** gives the truth value of ¬**A** corresponding to the truth value of **A** in the same row.

Conjunction

Another truth-functional operation about which little discussion is necessary is *conjunction.* We shall use **A** & **B** to stand for the conjunction (**A** and **B**). The truth table for & is

A	B	A & B
T	T	T
F	T	F
T	F	F
F	F	F

Fig. 1-2

There are four possible assignments of truth values to **A** and **B**. Hence there are four rows in the truth table. The only row in which **A** & **B** has the value T is the first row, where each of **A** and **B** is true.

Disjunction

The use of the word "or" in English is ambiguous. Sometimes, "**A** or **B**"† means that at least one of **A** and **B** is true, but that both **A** and **B** may be true. This is the *inclusive* usage of "or". Thus to explain someone's success one might say "he is very smart or he is very lucky", and this clearly does not exclude the possibility that he is both smart and lucky. The inclusive usage of "or" is often rendered in legal documents by the expression "and/or".

Sometimes the word "or" is used in an *exclusive* sense. For example, "Either I will go skating this afternoon or I will stay at home to study this afternoon" clearly means that I will not both go skating and stay home to study this afternoon. Whether the exclusive usage is intended by the speaker or is merely inferred by the listener is often difficult to determine from the sentence itself.

In any case, the ambiguity in usage of the word "or" is something that we cannot allow in a language intended for scientific applications. It is necessary to employ distinct symbols for the different meanings of "or", and it turns out to be more convenient to introduce a special symbol for the inclusive usage, since this occurs more frequently in mathematical assertions.††

"$\mathbf{A} \vee \mathbf{B}$" shall stand for "**A** or **B** or both". Thus in its truth table (Fig. 1-3) the only case where $\mathbf{A} \vee \mathbf{B}$ is false is the case where both **A** and **B** are false. The expression $\mathbf{A} \vee \mathbf{B}$ will be called a *disjunction* (of **A** and **B**).

A	**B**	$\mathbf{A} \vee \mathbf{B}$
T	T	T
F	T	T
T	F	T
F	F	F

Fig. 1-3

Conditionals

In mathematics, expressions of the form "If **A** then **B**" occur so often that it is necessary to understand the corresponding truth-functional operation. It is obvious that, when **A** is T and **B** is F, "If **A** then **B**" must be F. But in natural languages (like English) there is no established usage in the other cases (when **A** is F, or when both **A** and **B** are T). In fact when the meanings of **A** and **B** are not related (such as in "If the price of milk is 25¢ per quart, then high tide is at 8:00 P.M. today"), the expression "If **A** then **B**" is not regarded as having any meaning at all.

†Strictly speaking, we should employ quotation marks whenever we are talking about an expression rather than using it. However, this would sometimes get the reader lost in a sea of quotation marks, and we adopt instead the practice of omitting quotation marks whenever misunderstanding is improbable.

††In some natural languages, there are different words for the inclusive and exclusive "or". For example, in Latin, "vel" is used in the inclusive sense, while "aut" is used in the exclusive sense.

Thus if we wish to regard "If **A** then **B**" as truth-functional (i.e. the truth value must be determined by those of **A** and **B**), we shall have to go beyond ordinary usage. To this end we first introduce → as a symbol for the new operation. Thus we shall write "**A** → **B**" instead of "If **A** then **B**". **A** → **B** is called a *conditional* with *antecedent* **A** and *consequent* **B**. The truth table for → contains so far only one entry, in the third row.

A	B	A → B
T	T	
F	T	
T	F	F
F	F	

Fig. 1-4

As a guideline for deciding how to fill in the rest of the truth table, we can turn to "If (**C** & **D**) then **C**", which seems to be a proposition which should always be true. When **C** is T and **D** is F, (**C** & **D**) is F. Thus the second line of our truth table should be T (since (**C** & **D**) is F, **C** is T, and (If (**C** & **D**) then **C**) is T). Likewise when **C** is F and **D** is F, (**C** & **D**) is F. Hence the fourth line should be T. Finally, when **C** is T and **D** is T, (**C** & **D**) is T, and the first line should be T. We arrive at the following truth table:

A	B	A → B
T	T	T
F	T	T
T	F	F
F	F	T

Fig. 1-5

A → **B** is F when and only when **A** is T and **B** is F.

To make the meaning of **A** → **B** somewhat clearer, notice that **A** → **B** and (¬**A**) ∨ **B** always have the same truth value. (Just consider each of the four possible assignments of truth values to **A** and **B**.) Thus the intuitive meaning of **A** → **B** is "not-**A** or **B**". This is precisely the meaning which is given to "If **A** then **B**" in contemporary mathematics.

A proposition **A** → **B** is T whenever **A** is F, irrespective of the truth value of **B**. Notice also that **A** → **B** is automatically T whenever **B** is T, without regard to the truth value of **A**. In these two cases, one sometimes says that **A** → **B** is *trivially true* by virtue of the falsity of **A** or the truth of **B**.

Example 1.1.

The propositions $2+2=5 \rightarrow 1 \neq 1$ and $2+2=5 \rightarrow 1=1$ are both trivially true, since $2+2=5$ is false.

Biconditionals

At this time we shall introduce a special symbol for just one more truth-functional operation: **A** if and only if **B**. Let **A** ↔ **B** stand for "**A** if and only if **B**", where we understand the latter expression to mean that **A** and **B** have the same truth value (i.e. if **A** is T, so is **B**, and vice versa). This gives rise to the truth table:

A	**B**	**A ↔ B**
T	T	T
F	T	F
T	F	F
F	F	T

Fig. 1-6

A proposition of the form **A** ↔ **B** is called a *biconditional.* Notice that **A** ↔ **B** always takes the same truth value as (**A** → **B**) & (**B** → **A**); this is reflected in the mathematical practice of deriving a biconditional **A** ↔ **B** by proving **A** → **B** and **B** → **A** separately.

1.2 CONNECTIVES

Up to this point, we have selected five truth-functional operations and introduced special symbols for them: ¬, &, ∨, →, ↔. Of course if we limit ourselves only to two variables, then there are $2^4 = 16$ different truth-functional operations. With two variables, a truth table has four rows:

A	*B*	
T	T	–
F	T	–
T	F	–
F	F	–

Fig. 1-7

A truth-functional operation can have either T or F in each row. Hence there are $2 \cdot 2 \cdot 2 \cdot 2$ possible binary truth-functional operations.

Corresponding to any truth-functional operation (i.e. to any truth table) we can introduce a special symbol, called a *connective,* to indicate that operation. Thus the symbols ¬, &, ∨, →, ↔ are connectives. These five connectives will suffice for all practical purposes.

Example 1.2.

The operation corresponding to the exclusive usage of "or" could be designated by a connective +, having as its truth table:

A	**B**	**A + B**
T	T	F
F	T	T
T	F	T
F	F	F

Fig. 1-8

1.3 STATEMENT FORMS

To study the properties of truth-functional operations we introduce the following notions.

By a *statement form* (in the connectives ¬, &, ∨, →, ↔) we mean any expression built up from the *statement letters* $A, B, C, \ldots, A_1, B_1, C_1, \ldots$ by a finite number of applications of the connectives ¬, &, ∨, →, ↔. More precisely, an expression is a statement form if it can shown to be one by means of the following two rules:

(1) All statement letters (with or without positive integral subscripts) are statement forms.

(2) If **A** and **B** are statement forms, so are $(\neg \mathbf{A})$, $(\mathbf{A} \& \mathbf{B})$, $(\mathbf{A} \vee \mathbf{B})$, $(\mathbf{A} \rightarrow \mathbf{B})$, and $(\mathbf{A} \leftrightarrow \mathbf{B})$.†

Example 1.3.

Examples of statement forms:

(i) $(A \rightarrow (B \vee (C \& (\neg A))))$; (ii) $(\neg(A \leftrightarrow (\neg B_2)))$; (iii) $((\neg(\neg A_2)) \rightarrow (A_2 \rightarrow A_1))$.

Clearly we can talk about statement forms in any given set of connectives (instead of just $\neg$, &, $\vee$, $\rightarrow$. $\leftrightarrow$) by using the given connectives in clause (2) of the definition.

1.4 PARENTHESES

The need for parentheses in writing statement forms seems obvious. An expression such as $A \vee B \& C$ might mean either $((A \vee B) \& C)$ or $(A \vee (B \& C))$, and these two statement forms are not, in any sense, equivalent.

While parentheses are necessary, there are many cases in which some parentheses may be conveniently and unambiguously omitted. For that purpose, we adopt the following *conventions for omission of parentheses.*

(1) Every statement form other than a statement letter has an outer pair of parentheses. We may omit this outer pair without any danger of ambiguity. Thus instead of $((A \vee B) \& (\neg C))$, we write $(A \vee B) \& (\neg C)$.

(2) We omit the pair of parentheses around a denial $(\neg \mathbf{A})$. Thus instead of $(\neg A) \vee C$, we write $\neg A \vee C$. This cannot be confused with $\neg(A \vee C)$, since the parentheses will not be dropped from the latter. As another example consider $(A \& B) \vee (\neg(\neg(\neg B)))$. This becomes $(A \& B) \vee \neg\neg\neg B$.

(3) For any binary connective, we adopt the principle of *association to the left.* For example, $\mathbf{A} \& \mathbf{B} \& \mathbf{C}$ will stand for $(\mathbf{A} \& \mathbf{B}) \& \mathbf{C}$, and $\mathbf{A} \rightarrow \mathbf{B} \rightarrow \mathbf{C}$ will stand for $(\mathbf{A} \rightarrow \mathbf{B}) \rightarrow \mathbf{C}$.

Example 1.4.

Applying (1)-(3) above, the statement forms in the column on the left below are reduced to the equivalent expressions on the right.

$((\neg(\neg(A \& C))) \vee (\neg A))$	$\neg\neg(A \& C) \vee \neg A$
$((A \vee (\neg B)) \& (C \& (\neg A)))$	$(A \vee \neg B) \& (C \& \neg A)$
$(((A \vee (\neg B)) \& C) \& (\neg A))$	$(A \vee \neg B) \& C \& \neg A$
$((\neg A) \rightarrow (B \rightarrow (\neg(A \vee C))))$	$\neg A \rightarrow (B \rightarrow \neg(A \vee C))$

More far-reaching conventions for omitting parentheses are presented in Appendix A. In addition, Appendix B contains a method of rewriting statement forms so that no parentheses are required at all.

†An even more rigorous definition is: **B** is a statement form if and only if there is a finite sequence $\mathbf{A}_1, \ldots, \mathbf{A}_n$ such that

(1) $\mathbf{A}_n$ is **B**;

(2) if $1 \leq i \leq n$, then either $\mathbf{A}_i$ is a statement letter or there exist $j, k < i$ such that $\mathbf{A}_i$ is $(\neg \mathbf{A}_j)$ or $\mathbf{A}_i$ is $(\mathbf{A}_j \& \mathbf{A}_k)$ or $\mathbf{A}_i$ is $(\mathbf{A}_j \vee \mathbf{A}_k)$ or $\mathbf{A}_i$ is $(\mathbf{A}_j \rightarrow \mathbf{A}_k)$ or $\mathbf{A}_i$ is $(\mathbf{A}_j \leftrightarrow \mathbf{A}_k)$.

1.5 TRUTH TABLES

Every statement form **A** defines a truth-function: for every assignment of truth values to the statement letters in **A**, we can calculate the corresponding truth value of **A** itself. This calculation can be exhibited by means of a *truth table*.

Example 1.5.

The statement form $(\neg A \vee B) \leftrightarrow A$ has the truth table

A	B	$\neg A$	$\neg A \vee B$	$(\neg A \vee B) \leftrightarrow A$
T	T	F	T	T
F	T	T	T	F
T	F	F	F	F
F	F	T	T	F

Fig. 1-9

Each row corresponds to an assignment of truth values to the statement letters. The columns give the corresponding truth values for the statement forms occurring in the step-by-step construction of the given statement form.

Example 1.6.

The statement form $(A \vee (B \& C)) \rightarrow B$ has the truth table

A	B	C	$B \& C$	$A \vee (B \& C)$	$(A \vee (B \& C)) \rightarrow B$
T	T	T	T	T	T
F	T	T	T	T	T
T	F	T	F	T	F
F	F	T	F	F	T
T	T	F	F	T	T
F	T	F	F	F	T
T	F	F	F	T	F
F	F	F	F	F	T

Fig. 1-10

When there are three statement letters, notice that the truth table has eight rows. In general, when there are n statement letters, there are 2^n rows in the truth table, since there are two possibilities, T or F, for each statement letter.

Abbreviated Truth Tables

By the *principal connective* of a statement form (other than a statement letter), we mean the last connective used in the construction of the statement form. For example, $(A \vee B) \rightarrow C$ has $\rightarrow$ as its principal connective, $A \vee (B \rightarrow C)$ has $\vee$ as its principal connective, and $\neg(A \vee B)$ has $\neg$ as its principal connective.

There is a way of abbreviating truth tables so as to make the computations shorter. We just write down the given statement form once, and, instead of devoting a separate column to each statement form forming a part of the given statement form, we write the truth value of every such part under the principal connective of that part.

Example 1.7.

Abbreviated truth table for $(\neg A \vee B) \leftrightarrow A$. We begin with Fig. 1-11. Notice that each occurrence of a statement letter requires a repetition of the truth assignment for that letter.

$(\neg A$	$\vee$	$B)$	$\leftrightarrow$	A
T		T		T
F		T		F
T		F		T
F		F		F

Fig. 1-11

Then the negation is handled:

$(\neg$	A	$\vee$	$B)$	$\leftrightarrow$	A
F	T		T		T
T	F		T		F
F	T		F		T
T	F		F		F

followed by the disjunction

$(\neg$	A	$\vee$	$B)$	$\leftrightarrow$	A
F	T	T	T		T
T	F	T	T		F
F	T	F	F		T
T	F	T	F		F

and, finally, the biconditional

$(\neg$	A	$\vee$	$B)$	$\leftrightarrow$	A
F	T	T	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
T	F	T	F	F	F

Of course our use of four separate diagrams was only for the sake of illustration. In practice all the work can be carried out in one diagram.

1.6 TAUTOLOGIES AND CONTRADICTIONS

A statement form **A** is said to be a *tautology* if it takes the value T for all assignments of truth values to its statement letters. Clearly **A** is a tautology if and only if the column under **A** in its truth table contains only T's.

Example 1.8. $A \rightarrow A$ **is a tautology.**

A	$A \rightarrow A$
T	T
F	T

Fig. 1-12

Example 1.9. $A \vee \neg A$ **is a tautology.**

A	$\neg A$	$A \vee \neg A$
T	F	T
F	T	T

Fig. 1-13

Example 1.10. $(A \vee B) \leftrightarrow (B \vee A)$ is a tautology.

A	B	$A \vee B$	$B \vee A$	$(A \vee B) \leftrightarrow (B \vee A)$
T	T	T	T	T
F	T	T	T	T
T	F	T	T	T
F	F	F	F	T

Fig. 1-14

Example 1.11. $\underbrace{[A \,\&\, (B \vee C)] \leftrightarrow [(A \,\&\, B) \vee (A \,\&\, C)]}_{\mathbf{A}}$ is a tautology.†

A	B	C	$B \vee C$	$A \,\&\, (B \vee C)$	$A \,\&\, B$	$A \,\&\, C$	$(A \,\&\, B) \vee (A \,\&\, C)$	$\mathbf{A}$
T	T	T	T	T	T	T	T	T
F	T	T	T	F	F	F	F	T
T	F	T	T	T	F	T	T	T
F	F	T	T	F	F	F	F	T
T	T	F	T	T	T	F	T	T
F	T	F	T	F	F	F	F	T
T	F	F	F	F	F	F	F	T
F	F	F	F	F	F	F	F	T

Fig. 1-15

Theorem 1.1. If $\mathbf{K}$ is a tautology, and statement forms $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ are substituted for the statement letters $A, B, C, \ldots$ of $\mathbf{K}$ (the same statement form replacing all occurrences of a statement letter), then the resulting statement form $\mathbf{K}^{\#}$ is a tautology.

Example 1.12.

$(A \vee B) \leftrightarrow (B \vee A)$ is a tautology. Replace A by $(B \vee C)$ and simultaneously replace B by A. The new statement form $[(B \vee C) \vee A] \leftrightarrow [A \vee (B \vee C)]$ is a tautology.

Proof of Theorem 1.1. $\mathbf{K}$ determines a truth-function $f(A,B,C,\ldots)$ which always takes the value T no matter what the truth values of $A, B, C, \ldots$ may be. Let $g_1, g_2, g_3, \ldots$ be the truth-functions determined by $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$. Then the truth-function determined by $\mathbf{K}^{\#}$ must have the form $f^{\#} = f(g_1(\ldots), g_2(\ldots), g_3(\ldots), \ldots)$, and, since f always takes the value T, $f^{\#}$ also always takes the value T. ▶

A *contradiction* is a statement form which always takes the value F. Hence $\mathbf{A}$ is a contradiction if and only if $\neg\mathbf{A}$ is a tautology, and $\mathbf{A}$ is a tautology if and only if $\neg\mathbf{A}$ is a contradiction.

Example 1.13. $A \,\&\, \neg A$ is a contradiction.

A	$\neg A$	$A \,\&\, \neg A$
T	F	F
F	T	F

Fig. 1-16

†In writing this statement form, we have replaced some parentheses by brackets to improve legibility. For the same purpose, we also shall use braces.

Example 1.14. $A \leftrightarrow \neg A$ is a contradiction.

A	$\neg A$	$A \leftrightarrow \neg A$
T	F	F
F	T	F

Fig. 1-17

Example 1.15. $(A \vee B)\ \&\ \neg A\ \&\ \neg B$ is a contradiction.

A	B	$A \vee B$	$\neg A$	$\neg B$	$(A \vee B)\ \&\ \neg A$	$(A \vee B)\ \&\ \neg A\ \&\ \neg B$
T	T	T	F	F	F	F
F	T	T	T	F	T	F
T	F	T	F	T	F	F
F	F	F	T	T	F	F

Fig. 1-18

1.7 LOGICAL IMPLICATION AND EQUIVALENCE

We say that a statement form **A** *logically implies* a statement form **B** if and only if every assignment of truth values making **A** true also makes **B** true.

Example 1.16. **A** logically implies **A**.

Example 1.17. **A** logically implies **A** $\vee$ **B**. For, whenever **A** is true, **A** $\vee$ **B** also must be true.

Example 1.18. **A** & **B** logically implies **A**.

Theorem 1.2. **A** logically implies **B** if and only if **A** $\rightarrow$ **B** is a tautology.

Proof. **A** logically implies **B** if and only if, whenever **A** is true, **B** must also be true. Therefore **A** logically implies **B** if and only if it is never the case that **A** is true and **B** is false. But the latter assertion means that **A** $\rightarrow$ **B** is never false, i.e. that **A** $\rightarrow$ **B** is a tautology. ▶

Since we can effectively determine by a truth table whether a given statement form is a tautology, Theorem 1.2 provides us with an effective procedure for checking whether **A** logically implies **B**.

Example 1.19. Show that $(A \rightarrow B) \rightarrow A$ logically implies A.

Proof. Fig. 1-19 shows that $((A \rightarrow B) \rightarrow A) \rightarrow A$ is a tautology.

A	B	$A \rightarrow B$	$(A \rightarrow B) \rightarrow A$	$((A \rightarrow B) \rightarrow A) \rightarrow A$
T	T	T	T	T
F	T	T	F	T
T	F	F	T	T
F	F	T	F	T

Fig. 1-19

Statement forms **A** and **B** are called *logically equivalent* if and only if **A** and **B** always take the same truth value for any truth assignment to the statement letters. Clearly this means that **A** and **B** have the same entries in the last column of their truth tables.

Example 1.20. $A \leftrightarrow B$ is logically equivalent to $(A \to B) \& (B \to A)$.

A	B	$A \leftrightarrow B$	$A \to B$	$B \to A$	$(A \to B) \& (B \to A)$
T	T	T	T	T	T
F	T	F	T	F	F
T	F	F	F	T	F
F	F	T	T	T	T

Fig. 1-20

Theorem 1.3. **A** and **B** are logically equivalent if and only if **A** $\leftrightarrow$ **B** is a tautology.

Proof. **A** $\leftrightarrow$ **B** is T when and only when **A** and **B** have the same truth value. Hence **A** $\leftrightarrow$ **B** is a tautology (i.e. always takes the value T) if and only if **A** and **B** always have the same truth value (i.e. are logically equivalent). ▶

Example 1.21. $A \to (B \to C)$ is logically equivalent to $(A \& B) \to C$.

Proof. $[A \to (B \to C)] \leftrightarrow [(A \& B) \to C]$ is a tautology as shown in Fig. 1-21.

A	B	C	$B \to C$	$A \to (B \to C)$	$A \& B$	$(A \& B) \to C$	$[A \to (B \to C)] \leftrightarrow [(A \& B) \to C]$
T	T	T	T	T	T	T	T
F	T	T	T	T	F	T	T
T	F	T	T	T	F	T	T
F	F	T	T	T	F	T	T
T	T	F	F	F	T	F	T
F	T	F	F	T	F	T	T
T	F	F	T	T	F	T	T
F	F	F	T	T	F	T	T

Fig. 1-21

Corollary 1.4. If **A** and **B** are logically equivalent and we replace statement letters in **A** and **B** by statement forms (all occurrences of the same statement letter being replaced in both **A** and **B** by the same statement form), then the resulting statement forms are also logically equivalent.

Proof. This is a direct consequence of Theorems 1.3 and 1.1. ▶

Example 1.22.

$A \to (B \to C)$ and $(A \& B) \to C$ are logically equivalent. Hence so are $(C \vee A) \to (B \to (A \vee B))$ and $((C \vee A) \& B) \to (A \vee B)$ (and, in general, so are **A** $\to$ (**B** $\to$ **C**) and (**A** & **B**) $\to$ **C** for any statement forms **A**, **B**, **C**).

Theorem 1.5 (Replacement). If **B** and **C** are logically equivalent and if, within a statement form **A**, we replace one or more occurrences of **B** by **C**, then the resulting statement form **A**% is logically equivalent to **A**.

Proof. In the calculation of the truth values of **A** and **A**%, the distinction between **B** and **C** is unimportant, since **B** and **C** always take the same truth value. ▶

Example 1.23.

Let **A** be $(A \vee B) \to C$. Since $A \vee B$ is logically equivalent to $B \vee A$, **A** is logically equivalent to $(B \vee A) \to C$.

The following examples of logically equivalent pairs of statement forms will be extremely useful in the rest of this book, for the purpose of finding, for a given statement form, logically equivalent statement forms which are simpler or have a particularly revealing structure. We leave verification of their logical equivalence as an exercise.

Example 1.24. $\neg\neg\mathbf{A}$ and $\mathbf{A}$ (Law of Double Negation)

Example 1.25. (Idempotence)
(*a*) $\mathbf{A} \& \mathbf{A}$ and $\mathbf{A}$
(*b*) $\mathbf{A} \vee \mathbf{A}$ and $\mathbf{A}$

Example 1.26. (Commutativity)
(*a*) $\mathbf{A} \& \mathbf{B}$ and $\mathbf{B} \& \mathbf{A}$
(*b*) $\mathbf{A} \vee \mathbf{B}$ and $\mathbf{B} \vee \mathbf{A}$

Example 1.27. (Associativity)
(*a*) $(\mathbf{A} \& \mathbf{B}) \& \mathbf{C}$ and $\mathbf{A} \& (\mathbf{B} \& \mathbf{C})$
(*b*) $(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C}$ and $\mathbf{A} \vee (\mathbf{B} \vee \mathbf{C})$

As a result of the associative laws, we can leave out parentheses in conjunctions or disjunctions, if we do not distinguish between logically equivalent statement forms. For example, $A \vee B \vee C \vee D$ stands for $((A \vee B) \vee C) \vee D$, but the statement forms $(A \vee (B \vee C)) \vee D$, $A \vee ((B \vee C) \vee D)$, $(A \vee B) \vee (C \vee D)$ and $A \vee (B \vee (C \vee D))$ are logically equivalent to it.

Terminology: In $\mathbf{A}_1 \vee \mathbf{A}_2 \vee \cdots \vee \mathbf{A}_n$, the statement forms $\mathbf{A}_i$ are called *disjuncts*, while in $\mathbf{A}_1 \& \mathbf{A}_2 \& \ldots \& \mathbf{A}_n$ the statement forms $\mathbf{A}_i$ are called *conjuncts*.

Example 1.28. De Morgan's Laws.
(*a*) $\neg(\mathbf{A} \vee \mathbf{B})$ and $\neg\mathbf{A} \& \neg\mathbf{B}$
(*b*) $\neg(\mathbf{A} \& \mathbf{B})$ and $\neg\mathbf{A} \vee \neg\mathbf{B}$

Example 1.29. Distributive Laws (or Factoring-out Laws).
(*a*) $\mathbf{A} \& (\mathbf{B} \vee \mathbf{C})$ and $(\mathbf{A} \& \mathbf{B}) \vee (\mathbf{A} \& \mathbf{C})$
(*b*) $\mathbf{A} \vee (\mathbf{B} \& \mathbf{C})$ and $(\mathbf{A} \vee \mathbf{B}) \& (\mathbf{A} \vee \mathbf{C})$

Notice that there is a distributive law in arithmetic: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$; but the other distributive law, $a + (b \cdot c) = (a + b) \cdot (a + c)$ is false. (Take $a = b = c = 1$.)

Example 1.30. Absorption Laws.
(I) (*a*) $\mathbf{A} \vee (\mathbf{A} \& \mathbf{B})$ and $\mathbf{A}$
(*b*) $\mathbf{A} \& (\mathbf{A} \vee \mathbf{B})$ and $\mathbf{A}$
(II) (*a*) $(\mathbf{A} \& \mathbf{B}) \vee \neg\mathbf{B}$ and $\mathbf{A} \vee \neg\mathbf{B}$
(*b*) $(\mathbf{A} \vee \mathbf{B}) \& \neg\mathbf{B}$ and $\mathbf{A} \& \neg\mathbf{B}$
(III) If $\mathbf{T}$ is a tautology and $\mathbf{F}$ is a contradiction,
(*a*) $(\mathbf{T} \& \mathbf{A})$ and $\mathbf{A}$ (*c*) $(\mathbf{F} \& \mathbf{A})$ and $\mathbf{F}$
(*b*) $(\mathbf{T} \vee \mathbf{A})$ and $\mathbf{T}$ (*d*) $(\mathbf{F} \vee \mathbf{A})$ and $\mathbf{A}$

We shall often have occasion to use the logical equivalence between $(\mathbf{A} \& \neg\mathbf{B}) \vee \mathbf{B}$ and $\mathbf{A} \vee \mathbf{B}$, and between $(\mathbf{A} \vee \neg\mathbf{B}) \& \mathbf{B}$ and $\mathbf{A} \& \mathbf{B}$. We shall justify this by reference to Example 1.30(II), since it amounts to substituting $\neg\mathbf{B}$ for $\mathbf{B}$ in Example 1.30(II) and then using Example 1.24.

Example 1.31. $\mathbf{A} \rightarrow \mathbf{B}$ and $\neg\mathbf{B} \rightarrow \neg\mathbf{A}$ (Contrapositive)

Example 1.32. Elimination of conditionals.
(*a*) $\mathbf{A} \rightarrow \mathbf{B}$ and $\neg\mathbf{A} \vee \mathbf{B}$
(*b*) $\mathbf{A} \rightarrow \mathbf{B}$ and $\neg(\mathbf{A} \& \neg\mathbf{B})$