D. TER HAAR

ELEMENTS OF HAMILTONIAN MECHANICS

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BY

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PREFACE

For the last five years I have been giving a course of lectures on classical mechanics to last year undergraduates and first year research students at Oxford University. This course consisted of about 30 lectures and covered the subject matter of the present textbook. I devoted to each of the chapters roughly the same amount of time, namely about four hours, and feel that this particular course adequately covered those aspects of classical mechanics with which any physicist should be acquainted these days, without really delving into the many beautiful ramifications of which there are so many in classical mechanics. As the emphasis and the selection of subject matter in my course is rather different from the one in existing modern textbooks, of which those by Corben and Stehle and by Goldstein are the best known and most widely used examples, I felt that it might be of use to some people to publish this volume.

In writing this text I have benefited greatly from my own notes on lectures given by the late Professor H. A. Kramers at Leiden. I am also in debt to the various Oxford undergraduates whom I have taught classical mechanics. Finally, I should like to express my gratitude to Professor J. de Boer, Professor W. E. Lamb Jr., and Dr. W. E. Parry for critically reading through the manuscript of this volume and suggesting possible improvements.

D. ter Haar

Oxford, December 1960

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CHAPTER 1

NEWTONIAN MECHANICS

In this chapter we discuss briefly Newton's laws and apply them to the case of a central field of force with special reference to the inverse square law force. Scattering by a central field of force is briefly considered. Some properties of systems consisting of several particles which interact through two-body central forces are also discussed.

1. NEWTON'S LAWS

The basis of classical mechanics is formed by Newton's laws with which we shall start our discussion. We shall assume that the terms used in stating these laws have a well-defined meaning (they have all an intuitive meaning), as we do not wish to discuss the various concepts introduced here. In writing down Newton's laws we are assuming that there are systems of reference in which they are valid. Such systems are called *inertial systems*, and we shall assume that all our vectors are defined in such a system. We may add that any system moving with a uniform velocity with respect to an inertial system is again an inertial system.

We can now state Newton's laws:

Lex prima. If there are no forces acting on a particle it will persist in its motion, that is, it will move along a straight line with a constant velocity.

Lex secunda. If there are forces acting on a particle, the rate of change of the linear momentum of the particle will be equal to the total force acting on it. The linear momentum of a particle is defined as the product of its mass and its velocity.

Lex tertia. When two particles act upon one another, the force due to the first particle upon the second one is equal to, but in the opposite direction to, the force due to the second particle upon the first one. (*Actio est reactio*).

In using the term *particle* we shall throughout the book have in mind a point particle, that is, an entity characterised by its mass m, its position $\boldsymbol{x}$ and its velocity $\boldsymbol{v}$ which is equal to the rate of change of $\boldsymbol{x}$,

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}, \tag{1.101}$$

where t is the time coordinate.

As long as we do not discuss relativistic effects, we can assume m to be a constant which is characteristic of the particle.

We can express Newton's laws in mathematical form as follows:

Lex prima: If $\boldsymbol{F} = 0$, $\boldsymbol{v} = \text{constant}$; (1.102)

Lex secunda: $\dfrac{\mathrm{d}}{\mathrm{d}t} m\boldsymbol{v} = \boldsymbol{F}$; (1.103)

Lex tertia: $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$, or, $\boldsymbol{F}_{12}+\boldsymbol{F}_{21} = 0$. (1.104)

In these equations $\boldsymbol{F}$ is the total force on a particle, and $\boldsymbol{F}_{12}$ ($\boldsymbol{F}_{21}$) is the force exerted by particle number 2 (1) upon particle number 1 (2).

If m is constant, equation (1.103) can also be written in the form

$$m\boldsymbol{a} = \boldsymbol{F}, \qquad \boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}, \tag{1.105}$$

where $\boldsymbol{a}$ is the acceleration of the particle. This form of Newton's second law — force equals mass times acceleration — is the slightly more familiar one, but it is interesting to note that Newton himself used the other formulation which remains valid in the case of variable m.

The first law is Galileo's inertial law; hence the term inertial system. Comparing (1.105) and (1.102) we see that the first law is a special case (for $\boldsymbol{F} = 0$) of the second law. The mass m which can be considered to be defined by (1.103) is called the *inertial mass* of the particle; it has experimentally been shewn to be equal to the *heavy* (or gravitational) *mass* of a particle which is proportional to its weight. It may be mentioned here that this equality of the two kinds of mass follows naturally in the general theory of relativity.

Before discussing some of the consequences of Newton's laws we wish to mention that one sometimes finds as a lex quarta the rule that the addition of forces acting on a point particle follows the rules for the addition of vectors. This has tacitly been assumed in our equations (1.103) and (1.104), as we have used for the force a letter type typical of vectors.

Even without specifying the forces, we can draw some conclusions from (1.103) and (1.104). First of all, let us consider a system of two particles where the only forces acting upon the particles are $\boldsymbol{F}_{12}$ and $\boldsymbol{F}_{21}$. From (1.104) it follows that

$$\int_{t'}^{t''} (\boldsymbol{F}_{12}+\boldsymbol{F}_{21})\mathrm{d}t = 0. \tag{1.106}$$

As $\boldsymbol{F}_{12}$ ($\boldsymbol{F}_{21}$) is the only force acting upon the first (second) particle, we can use (1.103) to write

$$\boldsymbol{F}_{12} = \frac{\mathrm{d}m_1\boldsymbol{v}_1}{\mathrm{d}t}, \qquad \boldsymbol{F}_{21} = \frac{\mathrm{d}m_2\boldsymbol{v}_2}{\mathrm{d}t}, \tag{1.107}$$

and we get from (1.106) and (1.107)

$$[m_1\boldsymbol{v}_1+m_2\boldsymbol{v}_2]_{t'}^{t''} = 0, \tag{1.108}$$

or

$$\boldsymbol{p}_1'+\boldsymbol{p}_2' = \boldsymbol{p}_1''+\boldsymbol{p}_2'', \tag{1.109}$$

where the linear momentum $\boldsymbol{p}$ is defined by the equation

$$\boldsymbol{p} = m\boldsymbol{v}, \tag{1.110}$$

and where the primes (double primes) indicate the values of the quantities at t' (t'').

Equation (1.109) expresses the law of *conservation of (linear) momentum* which we have now shewn to hold for an isolated system of two interacting particles.

Let us now consider the motion of one particle under the influence of a force $\boldsymbol{F}$ and let us evaluate the integral

$$I = \int_{t'}^{t''} (\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{x}). \tag{1.111}$$

Using (1.103), and writing $\mathrm{d}\boldsymbol{x} = \boldsymbol{v}\mathrm{d}t$, we have

$$I = \int_{t'}^{t''} (m\ddot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}})\mathrm{d}t = [\tfrac{1}{2}m\dot{\boldsymbol{x}}^2]_{t'}^{t''} = T''-T', \tag{1.112}$$

where we have introduced the *kinetic energy* T by the equation

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}), \tag{1.113}$$

and where a dot (two dots) denote here and henceforth differentiating once (twice) with respect to the time.

As $(\boldsymbol{F}\cdot\boldsymbol{v})$ is the work done per unit time by the force on the particle, we see that (1.112) expresses the fact that the total work done by the forces acting upon a particle during a time interval (t', t'') is equal to the change in the kinetic energy of the particle. From (1.112) one can prove the *conservation of energy*, provided the field of force is a *conservative* one. A field of force is conservative, if the forces can be derived from a potential function U by the equation

$$\boldsymbol{F} = -\nabla U, \tag{1.114}$$

where ∇ is the gradient operator with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$.

In that case we have

$$\int_{t'}^{t''}(\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{x}) = -\int_{t'}^{t''}(\nabla U\cdot\mathrm{d}\boldsymbol{x}) = -U''+U', \tag{1.115}$$

or, using (1.111) and (1.112),

$$T'+U' = T''+U''. \tag{1.116}$$

The potential U is called the *potential energy* and we see from (1.116) that, provided U does not depend explicitly on t, the *total energy E*, that is, the sum of the kinetic and the potential energy,

$$E = T+U, \tag{1.117}$$

is a *constant of motion*, that is, does not change in time during the motion of the particle.

We note from (1.115) that in the case of a conservative field of force, the integral on the left hand side does not depend on the path of the particle, but only on its position at the beginning and at the end of the time interval considered. If this were not the case, we could not, of course, have introduced a potential function. Indeed, one can define a conservative field of force by the requirement that the integral I of (1.111) depends only upon the positions of the particle at t' and t'', but not on the path traversed between t' and t''.

If we are dealing with a one-dimensional conservative system, the equations of motion can always be solved by quadrature. Since the energy is a constant of motion, we can write

$$E = \tfrac{1}{2}m\dot{x}^2+U(x), \tag{1.118}$$

or,

$$\dot{x} = [(2/m)(E-U)]^{\frac{1}{2}}. \tag{1.119}$$

from which we get

$$t-t_0 = \int_{x_0}^{x} \{2[E-U(x)]/m\}^{-\frac{1}{2}}\mathrm{d}x, \tag{1.120}$$

where x_0 is the position of the particle at t_0.

A simple example of such a case is that of a linear harmonic oscillator, which is defined by a potential energy U given by the equation

$$U = \tfrac{1}{2}ax^2. \tag{1.121}$$

The solution (1.120) is now of the form

$$t-t_0 = (m/a)^{\frac{1}{2}} \arcsin [x(a/2E)^{\frac{1}{2}}],$$

or

$$x = (2E/a)^{\frac{1}{2}} \sin 2\pi\nu(t-t_0), \qquad 2\pi\nu = (a/m)^{\frac{1}{2}}, \tag{1.122}$$

where ν is the frequency of the harmonic oscillator, and where for the sake of simplicity we have put x_0 equal to zero.

We note in (1.122) that the energy is proportional to the square of the amplitude of the oscillation.

2. CENTRAL FIELDS OF FORCE

In many physical systems the forces are of a special kind, namely, central forces. These are forces which are acting along the line connecting the body on which the force is acting with the body producing the field of force. If we restrict our discussion to the case of a single particle in an external field of force, a central field of force is one in which the force acting on the particle will be directed along the line connecting the particle and a fixed point, the centre of the force field. If we choose the origin at the centre of the field, the force $\boldsymbol{F}$ acting on the particle will be of the form

$$\boldsymbol{F} = f(x, y, z)\boldsymbol{x}. \tag{1.201}$$

In general such a field of force need not be conservative. If it is conservative, the function $f(x, y, z)$ occurring in (1.201) can depend on the distance r from the origin only,

$$\boldsymbol{F} = f(r)\boldsymbol{x}. \tag{1.202}$$

This can be shewn by using (1.114) for a conservative force and evaluating the components of $\boldsymbol{F}$. We get

$$F_x = -\frac{\partial U}{\partial x}, \qquad F_y = -\frac{\partial U}{\partial y}, \qquad F_z = -\frac{\partial U}{\partial z}, \tag{1.203}$$

and as (1.201) must hold, we have in spherical polars r, θ, φ,

$$-\frac{\partial U}{\partial x} = fr\sin\theta\cos\varphi, \ -\frac{\partial U}{\partial y} = fr\sin\theta\sin\varphi, \ -\frac{\partial U}{\partial z} = fr\cos\theta, \tag{1.204}$$

from which it follows that

$$\frac{\partial U}{\partial r} = -fr, \qquad \frac{\partial U}{\partial \theta} = 0, \qquad \frac{\partial U}{\partial \varphi} = 0. \tag{1.205}$$

From the last two equations, it follows that U depends on r only, and from the first equation we then see that (1.202) must hold, while

$$U(\boldsymbol{x}) = U(r) = -\int^r rf(r)\mathrm{d}r. \tag{1.206}$$

Particular examples of central forces are the isotropic harmonic oscillator and the Coulomb or gravitational force field. In the first case the potential is given by the equation

$$U = \tfrac{1}{2}ar^2, \tag{1.207}$$

and in the second case by the equation

$$U = -\kappa/r. \tag{1.208}$$

If the constant κ in (1.208) is positive we are dealing with an attractive force, while a negative κ corresponds to a repulsive force. Equation (1.208) leads, of course, to an inverse square force, that is, a force proportional to the inverse square of the distance from the origin.

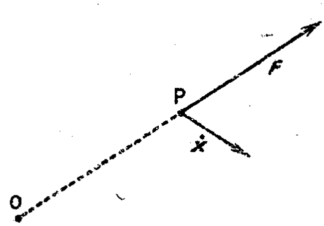

Fig. 1. Central field of force: O: centre of force field; P: position of particle; $\dot{\boldsymbol{x}}$: velocity vector of particle; $\boldsymbol{F}$: force acting upon particle.

The orbit of a particle on which the force (1.201) is acting will lie in a plane. This can be seen as follows (see fig. 1). According to Newton's second law, the acceleration of the particle will be in the direction of the force. Hence, both the acceleration and the velocity of the particle will be, and will remain, in the plane through the origin containing the velocity of the particle. Put differently, there will never be a component of the acceleration perpendicular to the plane through the origin and the velocity, so that the particle will stay in this plane, the *orbital plane.*

It is possible to prove that the orbit of a particle moving under the influence of a central field of force is a planar one by considering the *angular momentum* of the particle. The angular momentum with respect of the origin, $\boldsymbol{M}$, is defined by the equation

$$\boldsymbol{M} = [\boldsymbol{x} \wedge \boldsymbol{p}] = [\boldsymbol{x} \wedge m\dot{\boldsymbol{x}}]. \tag{1.209}$$

The angular momentum was in the past often called the moment of momentum, a very descriptive term. The nomenclature 'angular momentum' derives from the consideration of generalised coordinates which is given in the next chapter. The rate of change of $\boldsymbol{M}$ is given by the equation

$$\dot{\boldsymbol{M}} = [\dot{\boldsymbol{x}} \wedge m\dot{\boldsymbol{x}}]+[\boldsymbol{x} \wedge m\ddot{\boldsymbol{x}}] = 0+[\boldsymbol{x} \wedge f\boldsymbol{x}] = 0, \tag{1.210}$$

where we have used (1.105) and (1.201).

We have thus found for the case of a central force field a constant of motion, namely, $\boldsymbol{M}$, which is a vector, and thus really corresponds to three constants of motion. As $\boldsymbol{M}$ is constant, we see from (1.209) that the vector $\boldsymbol{x}$ will always lie in the fixed plane perpendicular to $\boldsymbol{M}$, which proves our statement that the particle orbit will be a planar one.

The problem of a particle in a central field of force can thus be reduced to a two-dimensional problem. The solutions of the original equations of motion, three second order differential equations, would contain six integration constants. Two of those can be chosen to be the direction cosines of $\boldsymbol{M}$, that is, they fix the normal to the orbital plane. In the remaining two-dimensional problem we are left with four integration constants which together with these two direction cosines make up the original six constants of motion. Of the remaining four, one will be the absolute magnitude of $\boldsymbol{M}$.

So far our conclusions are general and hold for forces which are not necessarily conservative. If the force field is conservative — and we shall from now on assume that this is the case — one of the last three constants of motion is the total energy and the other two will appear when the equations of motion are solved by quadrature, as can be done for a conservative central field.

Let us choose the z-axis of our system of coordinates along M and let us introduce polar coordinates r and θ in the xy-plane, that is, in the orbital plane, as follows

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1.211}$$

Expressed in terms of x and y the equations of motion are of the form

$$m\ddot{x} = -\frac{x}{r}\frac{\mathrm{d}U}{\mathrm{d}r}, \qquad m\ddot{y} = -\frac{y}{r}\frac{\mathrm{d}U}{\mathrm{d}r}, \tag{1.212}$$

where U is given by (1.206), and the force by (1.202).

If M be the absolute magnitude of the angular momentum and E the energy, we have from (1.209) and (1.117),

$$M = m(x\dot{y}-y\dot{x}), \qquad E = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)+U(r). \tag{1.213}$$

Equations (1.213) can, of course, be derived directly from (1.212) by quadrature.

Introducing r and θ we get instead of (1.213) the equations

$$M = mr^2\dot{\theta}, \tag{1.214}$$

$$E = \tfrac{1}{2}m\dot{r}^2+\tfrac{1}{2}mr^2\dot{\theta}^2+U(r). \tag{1.215}$$

Equation (1.214) describes the so-called *law of areas*. We know that M is a constant. To see the physical meaning of the right hand side of (1.214)

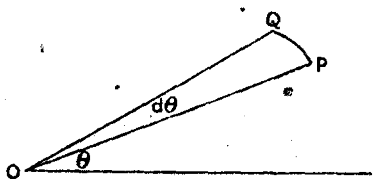

Fig. 2. The law of areas: O: centre of the force field; P: position of particle at time t; Q: position of particle at time $t+\mathrm{d}t$; θ: polar angle.

we consider fig. 2. Let the particle pass from P to Q during the time interval t, $t+\mathrm{d}t$. From the figure it is clear that during $\mathrm{d}t$ the radius vector sweeps through an area $\frac{1}{2}r^2\mathrm{d}\theta$. The areal velocity, defined as the area swept through per unit time, is thus equal to $\frac{1}{2}r^2\dot{\theta}$ and is constant according to (1.214). This is sometimes expressed as follows: The radius vector describes equal areas in equal times. For the case of the gravitational potential (1.208) this statement is known as Kepler's second law, but we have just seen that it is generally true for any central force, even for a non-conservative one.

Equation (1.215) gives an expression for the total energy in terms of polar coordinates, and we note that the kinetic energy T in planar polars is of the form

$$T = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2, \tag{1.216}$$

where the two terms on the right hand side refer respectively to the radial and transverse motion of the particle.

Eliminating $\dot{\theta}$ from (1.214) and (1.215) we get

$$E = \tfrac{1}{2}m\dot{r}^2 + U(r) + \frac{M^2}{2mr^2}. \tag{1.217}$$

The last term on the right hand side of this equation might be called the centrifugal potential energy. The absolute magnitude F_{cf} of the force corresponding to this term would be given by the equation

$$F_{cf} = -\frac{d}{dr}\frac{M^2}{2mr^2} = \frac{M^2}{mr^3} = \frac{mv_\perp^2}{r}, \tag{1.218}$$

where $v_\perp$ is the velocity component perpendicular to the radius vector ($M = mrv_\perp$). We see that we get for F_{cf} the usual expression for the centrifugal force acting on a particle which moves in a direction at an angle to the radius vector.

Formally (1.217) is of the same form as (1.118) for the one-dimensional case with an equivalent potential energy $U(r) + M^2/2mr^2$, that is, equal to the sum of the potential energy and centrifugal potential energy. We can thus solve this equation by quadrature, and the result is

$$t - t_0 = \int_{r_0}^{r} \frac{dr}{\left[\frac{2E}{m} - \frac{2U(r)}{m} - \frac{M^2}{m^2r^2}\right]^{\frac{1}{2}}}, \tag{1.219}$$

where r_0 is the value of r at t_0. Combining (1.219) with (1.214) we get

$$\theta - \theta_0 = \int_{t_0}^{t} \frac{M dt}{mr^2} = \int_{r_0}^{r} \frac{M dr}{r^2\{2m[E-U(r)] - M^2r^{-2}\}^{\frac{1}{2}}}. \tag{1.220}$$

This equation gives us θ as a function of r, that is, the form of the orbit. The remaining two integration constants are now fixed: they are the values of θ and r at t_0.

Before considering quantitatively the orbit for a special choice of U, namely (1.208), we shall consider qualitatively different kinds of orbits which arise when $U(r)$ behaves as r^{-1} both for very large and for very small values of r, although not necessarily with the same coefficient. Such a potential is of interest in atomic problems. The last electron in an atom will feel a potential $-e/r$ at large distances from the nucleus — the nuclear charge being

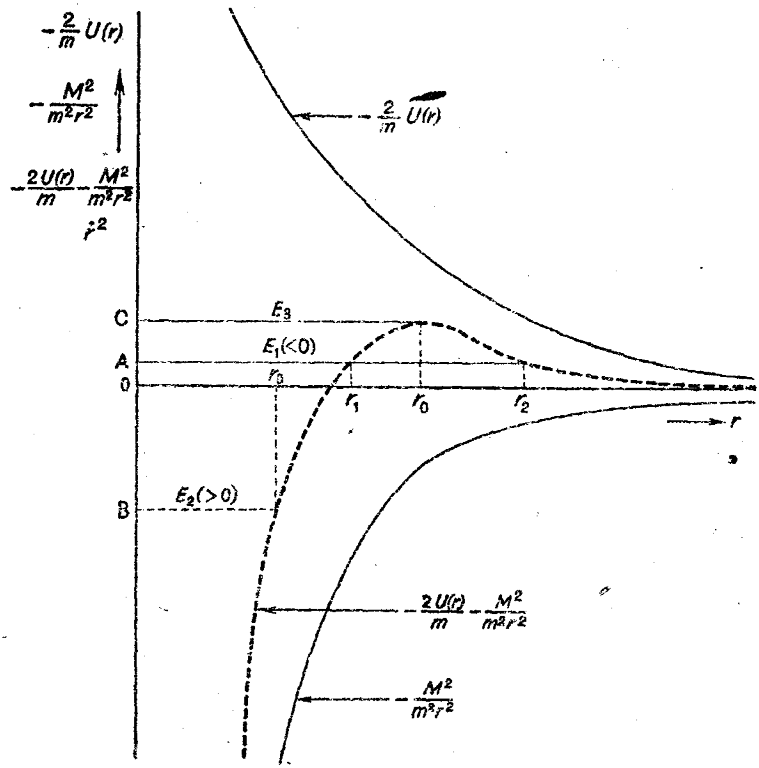

Fig. 3. Qualitative behaviour of orbits in a central potential.

a): Potential energy, centrifugal energy, and kinetic energy as functions of the distance from the centre of the force field. The $\dot{r}^2$-curve is the same as the $-(2/m)U(r)-M^2/m^2r^2$-curve, but with the abscissa axis starting from A, B, or C instead of O, depending on the value of E [A: E_1 (<0), B: E_2 (>0), C: E_3 (<0, circular orbit)].

screened by the other $Z-1$ electrons — and a potential $-Ze/r$ near the nucleus, when it is well inside the other electrons. In fig. 3 we have given first of all (Fig. 3a) $-2U(r)/m$, $-M^2/m^2r^2$, and their sum, as well as $\dot{r}^2$ as a function of r. This last function is obtained by rewriting (1.217) as follows

$$\dot{r}^2 = \frac{2E}{m} - \frac{2U(r)}{m} - \frac{M^2}{m^2r^2}; \qquad (1.221)$$

it must be noted that in fig. 3a the abscissa axis is different for different

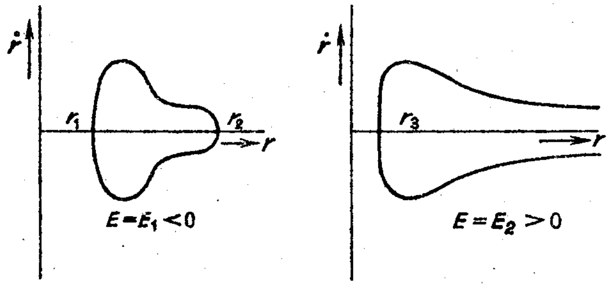

Fig. 3. b): Radial velocity as function of distance from centre of force field.

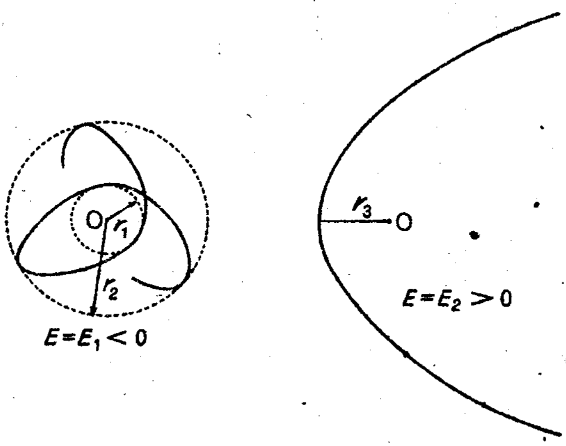

Fig. 3. c): Orbits for negative and positive energy values; O: centre of force field.

values of E in the $\dot{r}^2$ against r diagram. In fig. 3b we have plotted $\dot{r}$ as a function of r for two energy values E_1 and E_2 of which one is positive and the other one negative. It can be seen both from fig. 3a and from fig. 3b that for negative values of E the orbit will be one for which the particle does not disappear to infinity but is moving between two values of r, r_1 and r_2. It will thus in general be a rosette of the general shape shewn in fig. 3c. In the limiting case (E_3 in fig. 3a) where the r-axis just touches the $\dot{r}^2(r)$ curve we have a circular orbit. If E is positive, the orbit is an open one, and the particle has sufficient energy to disappear to infinity. This difference between open and closed orbits depending on whether E is positive or negative will crop up again presently, when we discuss the $1/r$-potential [see the discussion of (1.240)].

In many cases of interest the equations of motion simplify when we introduce $\sigma = r^{-1}$ instead of r as our coordinate. This substitution is the basis of Binet's method which is especially useful in the case where U is given by (1.208).

Let us first of all consider a decomposition of the force F acting on the particle into two components $F_{\parallel}$ and $F_{\perp}$ along and at right angles to the radius vector. We shall express this decomposition by using the Argand diagram of complex numbers,

$$m(\ddot{x}+i\ddot{y}) = (F_{\parallel}+iF_{\perp})e^{i\theta}. \tag{1.222}$$

If we write (1.211) in the composite form

$$x+iy = re^{i\theta}, \tag{1.223}$$

we find for the left hand side of equation (1.222)

$$m(\ddot{x}+i\ddot{y}) = m(\ddot{r}-r\dot{\theta}^2+ir\ddot{\theta}+2i\dot{r}\dot{\theta})e^{i\theta}, \tag{1.224}$$

and hence

$$F_{\perp} = m(r\ddot{\theta}+2\dot{r}\dot{\theta}) = \frac{m}{r}\frac{d}{dt}r^2\dot{\theta}, \tag{1.225}$$

$$F_{\parallel} = m(\ddot{r}-r\dot{\theta}^2). \tag{1.226}$$

As we are dealing with central forces, $F_{\perp}$ vanishes, and (1.225) leads to (1.214), while from equations (1.212) it follows that the left hand side of (1.224) is equal to $-(dU/dr)e^{i\theta}$, so that we can rewrite (1.226) in the form

$$F_{\parallel} = -\frac{dU}{dr} = m(\ddot{r}-r\dot{\theta}^2). \tag{1.227}$$