NONLINEAR ANALYSIS

A Collection of Papers in Honor of Erich H. Rothe

Edited by *LAMBERTO CESARI*

RANGACHARI KANNAN

HANS F. WEINBERGER

NONLINEAR ANALYSIS

A Collection of Papers in Honor of Erich H. Rothe

Edited by *LAMBERTO CESARI*
Department of Mathematics
University of Michigan
Ann Arbor, Michigan

RANGACHARI KANNAN
Mathematics Department
The University of Texas at Arlington
Arlington, Texas

HANS F. WEINBERGER
School of Mathematics
University of Minnesota
Minneapolis, Minnesota

ACADEMIC PRESS New York San Francisco London 1978
A Subsidiary of Harcourt Brace Jovanovich, Publishers

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1 7DX

Library of Congress Cataloging in Publication Data

Main entry under title:

Nonlinear analysis.

"Some of the papers in this book were presented at an international conference held in January 1969 at the University of the West Indies in Kingston, Jamaica."

Includes bibliographies and index.

CONTENTS: Amann, H. Periodic solutions of semilinear parabolic equations. --Brezis, H., and Browder, F. E. Linear maximal monotone operators and singular nonlinear integral equations of Hammerstein type. --Cesari, L. Nonlinear problems across a point of resonance for nonselfadjoint systems. [etc.]

1. Mathematical analysis--Addresses, essays, lectures. 2. Nonlinear theories--Addresses, essays, lectures. 3. Rothe, Erich H. I. Rothe, Erich H. II. Cesari, Lamberto. III. Kannan, Rangachary. IV. Weinberger, Hans F. V. Mona, Jamaica. University of the West Indies.

QA300.5.N66 515 77-6599
ISBN 0-12-165550-4

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

Professor Erich Rothe has made significant contributions to various aspects of nonlinear functional analysis. His early interests were in the field of parabolic and elliptic partial differential equations. Since then he has made fundamental contributions to the theory of nonlinear integral equations, gradient mappings, and degree theory. His more recent interests have been in critical point theory and the calculus of variations.

This volume is a collection of articles on nonlinear functional analysis dedicated to Professor Rothe on the occasion of his eightieth birthday. The intent of this collection is not to present a complete exposition of any particular branch of nonlinear analysis, but to provide an overview of some recent advances in the field.

CONTENTS

Periodic Solutions of Semilinear Parabolic Equations

Herbert Amann
Ruhr-University

Dedicated to Professor Erich H. Rothe on the occasion of his 80th birthday.

Introduction

In this paper we use some methods of nonlinear functional analysis, namely fixed-point theorems in ordered Banach spaces, to prove existence and multiplicity result for periodic solutions of semilinear parabolic differential equations of the second order.

The most natural and oldest method for the study of periodic solutions of differential equations is to find fixed points of the Poincaré operator, that is, the translation operator along the trajectories, which assigns to every initial value the value of the solution after one period (e.g., Krasnosel'skii [16]). In the case of parabolic equations it turns out that the Poincaré operator is compact in suitable function spaces. Moreover, by involving the strong maximum principle for linear parabolic equations, it can be shown that it is strongly increasing in some closed subspace of $C^{2+\nu}(\bar{\Omega})$, $0 < \nu < 1$.

This paper is motivated by some papers of Kolesov [12–14], who has used essentially the same approach. However, he considered the Poincaré operator in the space of continuous functions, and he did not realize that, even in the case of the general semilinear parabolic equations, this operator is strongly increasing. This latter fact is the basis for nontrivial existence and multiplicity results. For simplicity we present only one multiplicity result, namely we establish the existence of at least three periodic solutions, given certain conditions. But having shown that the Poincaré operator is strongly increasing, it is clear that we can put the problem in the general framework of nonlinear equations in ordered Banach spaces. Hence, by applying other general fixed-point theorems for equations of this type (e.g., [3,4,5,15]), it is possible to obtain further existence and multiplicity results.

We refer to the papers of Kolesov for further references. In addition we mention a paper by Fife [8], who, by different methods, obtained some existence theorems for periodic solutions of linear and quasi-linear parabolic equations. More recently, the work of Fife has been used by Bange [6] and Gaines and Walter [11] to obtain existence theorems in the case of one space variable. For further results on periodic solutions of nonlinear parabolic equations we refer to the references [7,10,20,23,26]. These authors use the theory of monotone operators to deduce existence theorems. However, none of these papers contains multiplicity results.

In the following section we introduce our hypotheses and present the main results. In Section 2 we collect some facts on abstract evolution equations in Banach spaces. Section 3 presents semilinear abstract evolution equations. It contains the basic a priori estimates, which, for further uses, are presented in somewhat greater generality than needed in this paper.

In Section 4 we study initial boundary value problems for semilinear parabolic differential equations. In particular we prove a global existence theorem (Theorem 4.5), which is of independent interest. In the last paragraph we establish the basic properties of the Poincaré operator and prove the existence and multiplicity results of Section 1.

1. Definitions and Main Results

Throughout this paper all functions are real-valued.

Let X and Y be nonempty sets with $X \subset Y$, and let $u: X \to \mathbb{R}$ and $v: Y \to \mathbb{R}$. Then we write $u \leq v$ if $u(x) \leq v(x)$ for every $x \in X$. If $u \leq v$ and $u \neq v \mid X$, then we write $u < v$. If $u > 0$, we say that u is positive, and if $u \geq 0$, it is called nonnegative.

We denote by Ω a bounded domain in $\mathbb{R}^N$, whose boundary Γ is an $(N-1)$-dimensional $C^{2+\mu}$-manifold for some $\mu \in (0, 1)$, such that Ω lies locally on one side of Γ.

We let

$$A(x, t, D)u := -\sum_{i,k=1}^{N} a_{ik}(x, t)D_iD_ku + \sum_{i=1}^{N} a_i(x, t)D_iu + a_0(x, t)u,$$

where (x, t) denotes a generic point of $\bar{\Omega} \times \mathbb{R}$. The coefficients a_{ik}, a_i, a_0 are supposed to be μ-Hölder continuous functions on $\bar{\Omega} \times \mathbb{R}$, where we use the metric $d((x, t), (y, s)) := (|x - y|^2 + |s - t|)^{1/2}$ for the computation of the Hölder constant (that is, a_{ik}, a_i, $a_0 \in C^{\mu, \mu/2}(\bar{\Omega} \times \mathbb{R})$). We assume that the coefficients of $A(x, t, D)$ are ω-periodic in t, for some $\omega > 0$, that $a_{ik} = a_{ki}$, and that there exists a positive constant μ_0 such that

$$\sum_{i,k=1}^{N} a_{ik}(x, t)\xi^i\xi^k \geq \mu_0 |\xi|^2,$$

for $(x, t) \in \bar{\Omega} \times \mathbb{R}$ and $\xi \in \mathbb{R}^N$. Hence

$$Lu := \frac{\partial u}{\partial t} + A(x, t, D)u$$

is a uniformly parabolic differential operator in $\bar{\Omega} \times \mathbb{R}$.

We denote by $\beta \in C^{1+\mu}(\Gamma, \mathbb{R}^N)$ an outward pointing, nowhere tangent vector field on Γ. Then we let $B = B(x, D)$ be a boundary operator on $\Gamma \times \mathbb{R}$ of the form

$$Bu := b_0 u + \delta \frac{\partial u}{\partial \beta},$$

where either $\delta = 0$ and $b_0 = 1$ (Dirichlet boundary operator), or $\delta = 1$ and $b_0 \in C^{1+\mu}(\Gamma)$ with $b_0 \geq 0$ (regular oblique derivative boundary operator). Observe that B is independent of t.

Let (x, t, ξ, η) be a generic point of $\bar{\Omega} \times \mathbb{R}^{N+2}$ with $x \in \bar{\Omega}$ and $\eta = (\eta^1, \ldots, \eta^N) \in \mathbb{R}^N$. Then we denote by $f: \bar{\Omega} \times \mathbb{R}^{N+2} \to \mathbb{R}$ a continuous function which is ω-periodic in t, such that $f(\cdot, \cdot, \xi, \eta): \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is μ-Hölder continuous, uniformly for (ξ, η) in bounded subsets of $\mathbb{R} \times \mathbb{R}^N$, and such that $\partial f/\partial \xi$ and $\partial f/\partial \eta^i$, $i = 1, \ldots, N$, exist and are continuous on $\bar{\Omega} \times \mathbb{R}^{N+2}$. Lastly, we suppose that there exist functions $c: \mathbb{R}'_+ \to \mathbb{R}_+ := [0, \infty)$ and $\varepsilon: \mathbb{R}_+ \to (0, 1)$ such that

$$|f(x, t, \xi, \eta)| \leq c(\rho)(1 + |\eta|^{2-\varepsilon(\rho)}), \tag{1.1}$$

for every $\rho \geq 0$ and $(x, t, \xi, \eta) \in \bar{\Omega} \times \mathbb{R} \times [-\rho, \rho] \times \mathbb{R}^N$.

Under the above assumptions we study the existence of ω-periodic solutions of the semilinear parabolic boundary value problem (BVP)

$$\begin{aligned} Lu &= f(x, t, u, \nabla u) \quad &&\text{in} \quad \Omega \times \mathbb{R}, \\ Bu &= 0 &&\text{on} \quad \Gamma \times \mathbb{R}. \end{aligned} \tag{1.2}$$

By an *ω-periodic solution* of the BVP (1.2) we mean a function $u \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$, which is ω-periodic in t, such that $Lu(x, t) = f(x, t, u(x, t), \nabla u(x, t))$ for $(x, t) \in \Omega \times \mathbb{R}$ and $Bu(x, t) = 0$ for $(x, t) \in \Gamma \times \mathbb{R}$, where $\nabla u = (D_1 u, \ldots, D_N u)$ denotes the gradient of u with respect to x. Of course, $u \in C^{2,1}$ means that u is continuously differentiable, twice with respect to x and once with respect to t. (In fact, it will be shown, that every ω-periodic solution of (1.2) belongs to $C^{2+\mu, 1+\mu/2}(\bar{\Omega} \times \mathbb{R})$.)

In the special case that the coefficients of $A(x, t, D)$ are independent of t (in which case we write $A(x, D)$), we can consider the linear elliptic eigenvalue problem (EVP)

$$A(x, D)u = \lambda u \quad \text{in } \Omega,$$
$$Bu = 0 \quad \text{on } \Gamma. \tag{1.3}$$

It is known (cf. Amann [3, Theorem 1.16]) that this EVP possesses a smallest eigenvalue λ_0. Moreover, $\lambda_0 > 0$ if $a_0 \geq 0$ and if, in the case that $\delta = 1$, $a_0 > 0$ for $b_0 = 0$.

After these preparations we can state an *existence and uniqueness theorem* for the linear case.

1.1 THEOREM Let one of the following hypotheses be satisfied:

(i) $a_0 \geq 0$. Moreover, if $\delta = 1$, then $a_0 > 0$ if $b_0 = 0$.

(ii) The coefficients of $A(x, t, D)$ are independent of t, and the smallest eigenvalue of the EVP (1.3) is positive.

Then, for every Hölder continuous function w on $\bar{\Omega} \times \mathbb{R}$, which is ω-periodic in t, the linear BVP

$$Lu = w \quad \text{in } \Omega \times \mathbb{R},$$
$$Bu = 0 \quad \text{on } \Gamma \times \mathbb{R},$$

has exactly one ω-periodic solution u, and $u > 0$ if $w > 0$.

A function u is called an *ω-subsolution* for the BVP (1.2) if there exists a number $T = T(u) > \omega$ such that $u \in C^{2,1}(\bar{\Omega} \times [0, T])$ and

$$Lu \leq f(\cdot, \cdot, u, \nabla u) \quad \text{in } \Omega \times (0, T],$$
$$Bu \leq 0 \quad \text{on } \Gamma \times (0, T],$$
$$u(\cdot, 0) \leq u(\cdot, \omega) \quad \text{on } \bar{\Omega}. \tag{1.4}$$

It is called a *strict ω-subsolution* if either $u(\cdot, 0) < u(\cdot, \omega)$ or $B(u(\cdot, 0)) < 0$. The notions of *ω-supersolutions* and *strict ω-supersolutions* are defined by reversing the above inequalities.

An ω-subsolution $\bar{v}$ and an ω-supersolution $\hat{v}$ are said to be *B-related*, if there exists a function $u \in C^{2+\mu}(\bar{\Omega})$ with $Bu = 0$, such that $\bar{v}(\cdot, 0) \le u \le \hat{v}(\cdot, 0)$. It can be shown (cf. Remark 5.5) that in the case of the first **BVP** (that is, if $\delta = 0$) every pair of ω-sub- and supersolutions $\bar{v}$, $\hat{v}$ satisfying $\bar{v}(\cdot, 0) \le \hat{v}(\cdot, 0)$ are B-related.

After these preparations we are ready for the statement of the main *existence theorem* for ω-periodic solutions of the semilinear parabolic BVP (1.2).

1.2 THEOREM Suppose that $\bar{v}$ is an ω-subsolution and $\hat{v}$ is an ω-supersolution for the BVP (1.2) such that $\bar{v}$ and $\hat{v}$ are B-related. Then there exists at least one ω-periodic solution u such that $\bar{v} \le u \le \hat{v}$.

More precisely, there exist a minimal ω-periodic solution $\bar{u}$ and a maximal ω-periodic solution $\hat{u}$ with $\bar{v} \le \bar{u} \le \hat{u} \le \hat{v}$, in the sense that $\bar{u} \le u \le \hat{u}$ for every ω-periodic solution u satisfying $\bar{v} \le u \le \hat{v}$.

In the case of the first BVP the above theorem is due to Kolesov [14] (cf. also Kolesov [12,13]). In fact, in Kolesov's theorem the coefficients of L are allowed to depend on u, and $\varepsilon(\rho)$ in (1.1) can be equal to zero (cf. Kolesov [14, Theorems 6 and 7]).

By combining Theorems 1.1 and 1.2, it is easy to give sufficient conditions for the existence of ω-periodic solutions.

1.3 THEOREM Suppose that there exist nonnegative Hölder continuous functions a and b on $\bar{\Omega} \times \mathbb{R}$, which are ω-periodic in t, such that

$$f(x, t, \xi, \eta) \le a(x, t)\xi + b(x, t),$$

for $(x, t, \xi, \eta) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^N$ and

$$f(x, t, \xi, \eta) \ge a(x, t)\xi - b(x, t), \tag{1.5}$$

for $(x, t, -\xi, \eta) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^N$. Moreover, suppose that the operator $L - a$ satisfies the hypotheses of Theorem 1.1. Then the BVP (1.2) has at least one ω-periodic solution.

Proof Theorem 1.1 implies that the BVP

$$\begin{aligned} (L - a)u = b \quad & \text{in} \quad \Omega \times \mathbb{R}, \\ Bu = 0 \quad & \text{on} \quad \Gamma \times \mathbb{R}, \end{aligned} \tag{1.6$_b$}$$

has exactly one ω-periodic solution $\hat{v}$, and $\hat{v} \ge 0$. Hence $\bar{v} := -\hat{v}$ is the unique ω-periodic solution of the BVP $(1.6)_{-b}$. It is clear that $\bar{v}$ is an ω-subsolution and $\hat{v}$ is an ω-supersolution which are B-related. Hence the assertion follows from Theorem 1.2. ●

1.4 COROLLARY Let the hypotheses of Theorem 1.3 be satisfied, but assume, instead of inequality (1.5), that $f(x, t, 0, 0) \geq 0$ for $(x, t) \in \bar{\Omega} \times \mathbb{R}$. Then the BVP (1.2) has at least one nonnegative ω-periodic solution.

Proof It suffices to observe that 0 is an ω-subsolution and that the solution $\hat{v}$ of $(1.6)_b$ is a nonnegative ω-supersolution, which is B-related to 0. ●

Corollary 1.4 generalizes considerably the results of Kolesov [14, Theorem 8], where it had been assumed that $\delta = 0$, and that f is nonnegative and independent of η.

In addition to the above existence results we prove the following *multiplicity theorem*, which is new, even in the simplest case that f is only a function of ξ.

1.5 THEOREM Suppose that $\bar{v}_1$ is an ω-subsolution, $\hat{v}_1$ is a strict ω-supersolution, $\bar{v}_2$ is a strict ω-subsolution, and $\hat{v}_2$ is an ω-supersolution, such that each one of the pairs $(\bar{v}_1, \hat{v}_1)$ and $(\bar{v}_2, \hat{v}_2)$ is B-related. Moreover, assume that $\hat{v}_1(\cdot, 0) \leq \bar{v}_2(\cdot, 0)$. Then the BVP (1.2) has at least three ω-periodic solutions u_j such that

$$\bar{v}_1 \leq u_1 < u_3 < u_2 \leq \hat{v}_2.$$

Moreover, $\bar{v}_j \leq u_j \leq \hat{v}_j$ for $j = 1, 2$, and $\bar{v}_2 \nleq u_3 \nleq \hat{v}_1$.

It should be remarked that, instead of $\hat{v}_1(\cdot, 0) \leq \bar{v}_2(\cdot, 0)$, it suffices to assume that $\bar{v}_1(\cdot, 0) < \hat{v}_2(\cdot, 0)$ and $\bar{v}_2(\cdot, 0) \nleq \hat{v}_1(\cdot, 0)$.

We close this section with a simple example. Consider the BVP

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= 4\pi^2 \cos u + u^2 a(x) \\ &\quad + e^u \sum_{k=1}^{N} a_k(x) \cos(kt)(D_k u)^{5/3} \qquad \text{in} \quad \Omega \times \mathbb{R}, \\ \frac{\partial u}{\partial \beta} &= 0 \qquad \text{on} \quad \Gamma \times \mathbb{R}, \end{aligned}$$

where the functions a and a_k are Hölder continuous on $\bar{\Omega}$ and $|a| \leq 1$. Then it is easily verified that the constant functions $\bar{v}_1 := -2\pi$, $\bar{v}_1 := -\pi$, $\bar{v}_2 := 0$, and $\hat{v}_2 := \pi$ satisfy the hypotheses of Theorem 1.2. Hence there exist at least three 2π-periodic solutions such that $-2\pi \leq u_1 < u_2 < u_3 \leq \pi$.

2. Preliminaries on Linear Evolution Equations

Throughout this paper all vector spaces are over the reals. If A is a linear operator in some Banach space, then we denote by $R(\lambda, A)$ the resolvent of the *complexification* of A. If X and Y are Banach spaces such that X is continuously imbedded in Y, then we write $X \hookrightarrow Y$.

Let X be a Banach space and let T be a fixed positive number. Suppose that

(A1) $\{A(t) \mid 0 \le t \le T\}$ is a family of closed densely defined linear operators in X such that the domain $D(A(t))$ of $A(t)$ is independent of t.

(A2) For each $t \in [0, T]$ the resolvent $R(\lambda, A(t))$ exists for all λ with $\operatorname{Re} \lambda \le 0$, and

$$\|R(\lambda, A(t))\| \le c(1 + |\lambda|)^{-1},$$

where c is some constant that is independent of λ and t.

These assumptions imply that $A(t)$ has an inverse $A^{-1}(t) \in L(X)$, where $L(X)$ denotes the Banach algebra of bounded linear operators on X. For abbreviation we write $A := A(0)$. Then $\|x\|_1 := \|Ax\|$ defines a norm on $D(A)$, which is equivalent to the graph norm. Consequently $X_1 := (D(A), \|\cdot\|_1)$ is a Banach space, and $X_1 \hookrightarrow X$. Moreover, by the closed graph theorem, $A(s)A^{-1}(t) \in L(X)$ for every $s, t \in [0, T]$. Hence $x \to \|A(s)x\|$ defines for each s an equivalent norm on $D(A)$, and $A(s) \in L(X_1, X)$, that is, $A(t)$ is a bounded linear operator from X_1 to X.

Using these notations, we suppose

(A3) The map $A(\cdot)\colon [0, T] \to L(X_1, X)$ is Hölder continuous.

In the following we denote by c, $c(\alpha, \ldots)$ generic constants, not necessarily the same in different formulas, which depend in an increasing way on the indicated quantities.

Assumptions (A1)–(A3) imply the existence of constants $c > 0$ and $\nu \in (0, 1)$ such that

$$\|(A(s) - A(t))A^{-1}(\tau)\| \le c|s - t|^{\nu} \tag{2.1}$$

for $s, t, \tau \in [0, T]$. In fact, it is easily seen that the map $\tau \to B(\tau) := A(\tau)A^{-1}$ is continuous from $[0, T]$ into the group $GL(X)$ of invertible operators in $L(X)$. Since the map $B \to B^{-1}$ is continuous on $GL(X)$, there exists a constant c such that $\|B^{-1}(\tau)\| = \|AA^{-1}(\tau)\| \le c$ for $\tau \in [0, T]$. Hence the assertion follows from the inequalities

$$\|(A(s) - A(t))A^{-1}(\tau)\| \le \|A(s) - A(t)\|_{L(X_1, X)} \|AA^{-1}(\tau)\| \le c|s - t|^{\nu}$$

for $s, t, \tau \in [0, T]$.

Assumptions (A1) and (A2) imply that $-A(t)$ is the infinitesimal generator of a holomorphic semigroup $\{e^{-\tau A(t)} \mid 0 \le \tau < \infty\}$ in $L(X)$. Moreover, there exist positive constants c and δ_0 such that

$$\|e^{-\tau A(t)}\| \le ce^{-\delta_0 \tau} \tag{2.2}$$

and

$$\|A(t)e^{-\tau A(t)}\| \leq c\tau^{-1}e^{-\delta_0\tau} \tag{2.3}$$

for $\tau > 0$ and $t \in [0, T]$.

Then inequality (2.2) implies the existence of the integral

$$A^{-\alpha}(t) := \frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} e^{-\tau A(t)}\, d\tau \tag{2.4}$$

for every $\alpha > 0$. It follows that $A^{-1}(t) = [A(t)]^{-1}$, and each $A^{-\alpha}(t)$ is an injective continuous endomorphism of X. Hence $A^\alpha(t) := [A^{-\alpha}(t)]^{-1}$ is a closed bijective linear operator in X. It can be shown that each $A^\alpha(t)$ has dense domain and that $D(A^\alpha(t)) \subset D(A^\beta(t))$ for $a \geq \beta \geq 0$. Moreover,

$$A^{\alpha+\beta}(t)x = A^\alpha(t)A^\beta(t)x = A^\beta(t)A^\alpha(t)x$$

for every $\alpha, \beta \in \mathbb{R}$ and $x \in D(A^\gamma(t))$, with $\gamma := \max\{\alpha, \beta, \alpha + \beta\}$, where $A^0(t) = id_X$. (For proofs of these facts we refer to the literature [9,17,21,24].)

It has been shown by Sobolevskii [24, inequality (1.59)] that $D(A^\beta(s)) \subset D(A^\alpha(t))$ for $0 \leq \alpha < \beta \leq 1$ and $s, t \in [0, T]$, and that

$$\|A^\alpha(s)A^{-\beta}(t)\| \leq c(\alpha, \beta) \tag{2.5}$$

for $s, t \in [0, T]$.

In the following we let $\|x\|_\alpha := \|A^\alpha x\|$ for $x \in D(A^\alpha)$ and $0 \leq \alpha \leq 1$, and we denote by X_α the Banach space $(D(A^\alpha), \|\cdot\|_\alpha)$. Then $X_\beta \hookrightarrow X_\alpha$ for $0 \leq \alpha \leq \beta \leq 1$ (with $X_0 = X$).

We consider the linear initial value problem (IVP)

$$\begin{aligned} u' + A(t)u &= g(t), \qquad 0 < t \leq T, \\ u(0) &= x, \end{aligned} \tag{2.6}$$

with $g \in C([0, T], X)$ and $x \in X$. By a *solution* u of (2.6) we mean a function $u \in C([0, T], X) \cap C^1((0, T], X)$ with $u(0) = x$, $u(t) \in D(A)$ for $t > 0$, and $u'(t) + A(t)u(t) = g(t)$ for $0 < t \leq T$.

Our assumptions (cf. inequality (2.1)) imply that the results of Sobolevskii [24] and Tanabe [25] are applicable (cf. also Friedman [10]). Hence the IVP (2.6) has a unique solution u for every Hölder continuous right-hand side g. Moreover, $u \in C^1([0, T], X)$, provided $x \in D(A)$.

Sobolevskii and Tanabe have shown that there exists a unique evolution operator $U(t, \tau) \in \mathcal{L}(X)$, $0 \leq \tau \leq t \leq T$, such that every solution u of the IVP (2.6) can be represented in the form

$$u(t) = U(t, 0)x + \int_0^t U(t, \tau)g(\tau)\, d\tau, \qquad 0 \leq t \leq T. \tag{2.7}$$

The function U is strongly continuous on the closure of the set $\Delta := \{(t, \tau) \in [0, T]^2 \mid 0 \le \tau < t \le T\}$ (that is, $U(\cdot)x \in C(\bar{\Delta}, X)$ for every $x \in X$) and satisfies $U(t, t) = id_X$, $U(s, t)U(t, \tau) = U(s, \tau)$ for $0 \le \tau \le t \le s \le T$. Moreover, U has an important smoothing property, namely $U(s, t)X \subset D(A)$ for $0 \le t < s \le T$.

In the following lemma we collect the most important regularity properties of the evolution operator. For abbreviation we denote the norm in $L(X_\alpha, X_\beta)$ by $\|\cdot\|_{\alpha,\beta}$.

2.1 **Lemma** (i) Suppose that $0 \le \alpha \le \beta < 1$. Then

$$\|U(t, \tau)\|_{\alpha,\beta} \le c(\alpha, \beta, \gamma)(t - \tau)^{-\gamma} \tag{2.8}$$

for $\beta - \alpha < \gamma < 1$ and $0 \le \tau < t \le T$. Moreover, if $0 \le \beta < \alpha \le 1$, then

$$\|U(t, \tau)\|_{\alpha,\beta} \le c(\alpha, \beta). \tag{2.9}$$

(ii) Suppose that $0 \le \alpha < \beta \le 1$. Then

$$\|U(t, \tau) - U(s, \tau)\|_{\beta,\alpha} \le c(\alpha, \beta, \gamma)\,|t - s|^\gamma \tag{2.10}$$

for $0 \le \gamma < \beta - \alpha$ and $(t, \tau), (s, \tau) \in \bar{\Delta}$.

(iii) Let $0 \le \alpha < 1$, $0 \le \sigma < T$, and $g \in C([\sigma, T], X)$. Then

$$\left\| \int_\sigma^t U(t, \tau)g(\tau)\, d\tau - \int_\sigma^s U(s, \tau)g(\tau)\, d\tau \right\|_\alpha \le c(\alpha, \gamma)\,|s - t|^\gamma \max_{\sigma \le \tau \le T} \|g(\tau)\| \tag{2.11}$$

for $0 \le \gamma < 1 - \alpha$ and $\sigma \le s, t \le T$.

Proof (i) Suppose that $\alpha > 0$ and let

$$0 < \varepsilon < \min\{\alpha, 1 - \beta, (\gamma - \beta + \alpha)/2\}.$$

Then

$$\begin{aligned}\|U(t, \tau)\|_{\alpha,\beta} &\le \|A^\beta U(t, \tau)A^{-\alpha}\| \\ &\le \|A^\beta A^{-\beta-\varepsilon}(t)\|\ \|A^{\beta+\varepsilon}(t)U(t, \tau)A^{-\alpha+\varepsilon}(\tau)\|\ \|A^{\alpha-\varepsilon}(\tau)A^\alpha\|.\end{aligned}$$

Hence it follows from (2.5) and Sobolevskii [24, inequality (1.65)] (cf. also Friedman [10, inequality (II.14.12)]) that

$$\|U(t, \tau)\|_{\alpha,\beta} \le c(\alpha, \beta, \varepsilon)(t - \tau)^{\alpha-\beta-2\varepsilon}.$$

This implies (2.8).

If $\alpha = 0$, then the estimate (2.8) follows in a similar way from

$$\|U(t, \tau)\|_{0,\beta} \le \|A^\beta A^{-\gamma}(t)\|\ \|A^\gamma(t)U(t, \tau)\|.$$