Studies in Applied Mathematics

A VOLUME DEDICATED TO IRVING SEGAL

ADVANCES IN MATHEMATICS
SUPPLEMENTARY STUDIES, VOLUME 8

EDITED BY

Victor Guillemin

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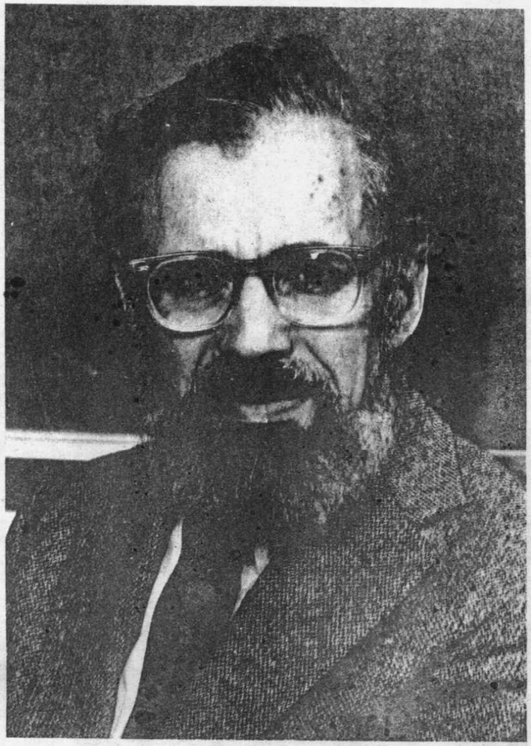

Irving Segal

Preface

The contents of this volume are a testament to the continuing vitality of functional analysis as a clearinghouse for ideas in the various "hard" disciplines of modern analysis, e.g., harmonic analysis on Lie groups, spectral theory and scattering theory, and the qualitative theory of nonlinear PDEs. Functional analysis has, of course, always played this role ever since its inception in the early 1900s in the work of Hilbert, Banach, Frechet, Riesz, and Von Neumann. It began to play this role much more decisively, however, in the early 1940s, when Gelfand and Segal began publishing their papers on locally compact groups, Wigner and Mackey created the theory of induced representations, Bergman's work on representations of the Lorentz group set the stage for the later work of Harish-Chandra, and Schauder and Leray published their seminal papers on nonlinear elliptic PDEs. To a certain extent subsequent developments in all the areas alluded to above have been no more than an effort to digest the developments of this period. An example is the "abstract" Plancherel formula of Segal for unimodular locally compact groups. This theorem provides one, a priori, with the Plancherel measure, but the problem of actually writing it down in concrete cases was not settled until decades later with the work of Harish-Chandra in the semisimple case and of Kirillov and Auslander–Kostant in the solvable case.

For the most part the contents of this volume do not deal with the machinery of functional analysis per se, but with the applications. (An exception is Sakai's paper on C^*-algebras. One of the more agreeable surprises of the past ten years has been the renewed life pumped into this subject by Alain Connes and others.) The applications range from the extremely pure to the very applied and touch on subjects as unrelated as constructive field theory and fluid dynamics. However, by a small miracle, virtually any trained mathematician of the present generation can read most of these articles, with some benefit, thanks to the language of functional analysis, the lingua franca of present-day mathematics, in which they are written. I have described this as a small miracle because without this language mathematicians in different specialities would nowadays be communicating across barriers far more formidable than those that impeded interdisciplinary communication 75 years ago.

Many mathematicians (an adequate list would be impossible to provide in this preface) have played a role in the history sketched above. A few mathematicians have played a role in *all* phases of this history. Prominent among them is Irving Segal, and it is fitting that this volume be dedicated to him.

Introduction

Irving Ezra Segal was born in New York on September 13, 1918, went to public schools in Trenton, was an undergraduate at Princeton, and received his Ph.D. in mathematics at Yale in 1940. Einar Hille was his thesis adviser. After some time at Harvard and Princeton Universities and The Institute for Advanced Study, he joined the faculty of the University of Chicago in 1948, and in 1960 he moved to Massachusetts Institute of Technology. During this time he has produced 136 papers and five books, and has supervised 37 Ph.D. dissertations.

A summary of Segal's thesis [3]† was published in 1941. War work and military service delayed a fuller account [7] until 1947. This thesis was a major work which showed the power of abstract functional analysis in extending and clarifying classical results of harmonic analysis. At the time this work was done it would have appeared unlikely that maximal ideals, whose existence was proved by transfinite induction, could be identified with concrete objects (characters in the commutative case). Segal was one of the pioneers of new methods in analysis, proving the existence of analytical objects by characterizing them from above by general properties rather than directly constructing them from below. This line of work was pursued in a number of papers, including [12], which characterized closed two-sided ideals in C^*-algebras, and [13] and [15], which are fundamental to the harmonic analysis of a general locally compact unimodular group.

An important paper from this period is his system of postulates for general quantum mechanics [9], which appeared in 1947. This paper introduced the important concept that it is the observables and their states, not the underlying Hilbert space, which are the key features of quantum theory, an idea which he developed later in a number of papers including [34]. This viewpoint is so widespread now that it is hard to realize that it appeared radical at the time. Equally important

† Numbers in brackets [ ] refer to Irving Segal's journal articles and numbers in braces { } to his books. To keep this introduction manageable, there are no references to the work of others.

to probability theory was the related insight [22] that it is the algebra of random variables, not the underlying probability space, which is essential. This point of view led to beautiful results on second quantization in [25], [26], and [31], and the whole development is reviewed in [50].

In 1960 Segal published [38] a framework for the development of a theory of interacting (nonlinear) quantum fields. As is characteristic of his work, rather than attempting to solve technical problems within an accepted framework, Segal asks first what is essential in the framework, what is extraneous, and what is a proper general setting. Although the flowering of constructive quantum field theory took place largely outside of Segal's proposed framework, it remains as a challenge to find a deeper understanding of the equations of motion of a quantized nonlinear field. Segal's ideas on nonlinear quantum fields are highly geometrical and have stimulated beautiful work by him (beginning with [41] and [42] and continuing through many later papers) and his students on classical nonlinear equations and the geometrical structure of their infinite-dimensional solution space.

In 1951 Segal published a paper [17] full of novel mathematical and physical ideas. In particular, he showed that the Lie algebra of the Poincaré (inhomogeneous Lorentz) group is a limiting case (or contraction as the notion later became known among physicists) of the Lie algebra of the conformal group. Just as classical mechanics is a limiting case of quantum mechanics when $\hbar \to 0$, and nonrelativistic mechanics is a limiting case of relativistic mechanics when $c \to \infty$, so might relativistic quantum theory be a limiting case of a more general theory as the structure constants of the conformal Lie algebra tend to those of the Poincaré Lie algebra. Such a theory would agree with the usual theory at familiar distance scales, but would be very different at extremely large and at extremely small distances. In the past decade Segal has vigorously pursued the large-distance consequences of this idea {5}. The result is a cosmology totally at variance with the usual expanding universe cosmologies. Segal's book is an unusual blend of mathematical theory and analysis of observational data, and it shows how precarious a foundation is Hubble's law for the elaborate superstructure that has been erected upon it.

I have briefly touched on some of the main themes in Segal's work which show why it is unique and exciting. He is also a unique and exciting teacher. Let me now speak in a personal vein, and perhaps I shall to some extent speak for his other students as well. Irving Segal is a constant source of fresh ways of looking at things. I have frequently observed him in conversations with experts in differential geometry, partial differential equations, group representations, probability theory, etc. He will pose a disturbing question. A frequent reaction will be an unspoken feeling that that is not the kind of question one asks in this field and a puzzled sense that there may be something quite interesting involved. I took a course in probability theory from him at the University of Chicago, and the subject came alive with a special viewpoint, a sense of what is basic, and prospects for research. His encouragement was strong when I was writing a thesis, and equally important was his total lack of encouragement when I found a result unrelated to anything beyond itself. One of the chief characteristics of Segal's work is that his theorems are part of theories, and this sense of the global nature of mathematical research is one of the most valuable things that he imparts to his students.

Mathematicians tend to be mistrustful of the world, except as a source of problems. Physicists tend to be mistrustful of mathematics, except as a tool. Irving Segal is a mathematician with a passionate conviction that mathematics can give us knowledge about the world. This conviction is so strong that it led him from a discovery about Lie algebras in 1951 to years of work in the 1970s analyzing data on galaxies and quasars; it is so strong that it has often involved him in scientific controversies. In some of these controversies his vision of what form the theory should take has on occasion led him to give less value to the work of others than I for one would give; certainly his own work has frequently been unfairly depreciated by established experts. We are fortunate that controversy in our field concerns matters that can be tested, and that it can lead to an increase in knowledge.

Irving Segal once remarked that his original intention was to go into class field theory. It is safe to say that had he done so, that kind of field theory would have received from him as strong a personal imprint as this kind of field theory bears today: fundamental viewpoints due to him and later taken for granted by workers in the field, a large-scale vision of how the theory should develop, the discovery of deep results and unsuspected connections, and provocative challenges to the accepted wisdom.

Edward Nelson

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STUDIES IN APPLIED MATHEMATICS
ADVANCES IN MATHEMATICS SUPPLEMENTARY STUDIES, VOL. 8

A Remark on the Polymer Problem in Four Dimensions†

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DEDICATED TO IRVING SEGAL

Constructive quantum field theory has achieved great success in studying superrenormalizable models, but the barrier that separates them from the renormalizable ones has not yet been penetrated. The purpose of this chapter is to focus attention on some aspects of the simplest renormalizable problem of all: the polymer problem in four dimensions.

A polymer is a long object whose thickness is extremely small in comparison to its length. Yet the thickness has important consequences in the excluded volume effect: a polymer may not loop back and cross itself. It is the excluded volume effect that keeps a plateful of spaghetti from lying flat on the plate.

What does a typical polymer look like? Were it not for the excluded volume effect we could model polymers by Wiener paths. Edwards [3] proposed to model polymers by introducing a weighting factor on Wiener paths, formally given by

$$N\exp\left(-g\int_0^1\int_0^1 \delta(\omega(s)-\omega(t))\,ds\,dt\right), \tag{1}$$

where δ is the Dirac delta function, g is a coupling constant measuring the strength of the excluded volume effect, and N is a normalization constant.

More precisely (cf. [7]), let Ω be the space of all functions from $[0, 1]$ to $\mathbb{R}^d$ and let ν be the Wiener measure on Ω for a given variance parameter σ^2. Let δ_ε be a smooth positive function on $\mathbb{R}^d$ that tends to δ, in the sense of distributions, as $\varepsilon \to 0$. Let μ_ε be the probability measure

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on Ω whose Radon–Nikodym derivative with respect to ν is

$$N_\varepsilon \exp\left(-g_\varepsilon \int_0^1 \int_0^1 \delta_\varepsilon(\omega(s) - \omega(t))\, ds\, dt \right), \tag{2}$$

where g_ε is an arbitrary positive function of ε and the variance parameter σ^2 of ν is also allowed to be an arbitrary positive function of ε. Then any probability measure μ that is a weak limit of μ_ε as $\varepsilon \to 0$ is called a *polymer measure.*

Polymer measures also occur in the seminal paper by Symanzik [5] on Euclidean field theory. In this paper Symanzik showed that the Euclidean theory corresponding to a scalar quantum field with a φ^4 self-interaction is the classical statistical mechanics of polymers with a local interaction. In the classical statistical mechanics of a gas, a configuration of point molecules is given the weighting factor

$$\exp[-\beta \sum U(x_i - x_j)]. \tag{3}$$

In Symanzik's theory (simplified for purposes of exposition) a configuration of polymers is given the weighting factor

$$\exp\left(-g \sum \int_0^1 \int_0^1 \delta(\omega_i(s) - \omega_j(t))\, ds\, dt \right). \tag{4}$$

There is one important difference: in the classical statistical mechanics of a gas the terms $i = j$ in (3) give only a constant and may be omitted, whereas in Symanzik's theory the terms $i = j$ in (4) are essential. Symanzik's theory is not a theory of interacting Wiener paths; it is a theory of interacting polymers.

There are deep results on polymer measures in all interesting dimensions except for $d = 4$. For $d = 2$, Varadhan, in the Appendix to [5], showed that

$$\int_0^1 \int_0^1 \delta_\varepsilon(\omega(s) - \omega(t))\, ds\, dt - E \int_0^1 \int_0^1 \delta_\varepsilon(\omega(s) - \omega(t))\, ds\, dt,$$

where E denotes the expectation (integral with respect to ν), converges as $\varepsilon \to 0$ to a random variable, which is in L^p for all $p < \infty$, giving a polymer measure that is absolutely continuous with respect to Wiener measure. For $d = 3$ Westwater [7] proved the existence of a nontrivial polymer measure for g sufficiently small. In both cases the methods were based on techniques of Euclidean field theory, going in the opposite direction from Symanzik's proposal. The original Symanzik method has never been carried through, in any dimension, to construct a field theory (but see [8] for related results). This would be a worthwhile project. For $d \geq 5$ Lawler [4] has shown that the self-avoiding random walk obtained by erasing, as they occur, all closed loops in an unrestricted random walk on a lattice converges, as the lattice spacing tends to zero, to a Wiener process. This result of Lawler's gives strong support to, but does not prove, the conjecture that every polymer measure in dimension $d \geq 5$ is just a Wiener measure.

The measure that Westwater constructs is singular with respect to Wiener measure. Little is known about the properties of the sample paths of the Westwater process. For example, is a Westwater path bigger or smaller (say, in terms of exact Hausdorff dimension) than a Wiener path? One intuition is that it should be smaller because it has no loops. Symanzik, however, gives a convincing heuristic argument [5, pp. 201–202] that it should be bigger. Now that Westwater has constructed the measure, it should be possible to make this argument into a theorem for $d = 3$. With probability one, a Wiener path in dimension $d \geq 2$ has Hausdorff dimension 2 but has zero Hausdorff 2-measure (see [6] and [1] for the exact Hausdorff dimension of Wiener paths in dimension 2 and in dimension $d \geq 3$, respectively). Does a Westwater path have nonzero Hausdorff 2-measure with probability one?

The time it takes a Wiener path to go distance ε is of order ε^2. Let τ be a function of ε such that $\varepsilon^2/\tau \to 0$ as $\varepsilon \to 0$, and let r be a fixed time. A pair $\langle s, t \rangle$ with $s \leq r \leq t$ such that $|\omega(s) - \omega(t)| \leq \varepsilon$ and $t - s \geq \tau$ will be called a *near loop* about r. For small ε, the time τ is enormous compared to ε^2, so in general a near loop of a Wiener path will travel an enormous multiple of ε away before coming back to within ε of itself. Now suppose we have another function $\tilde{\tau}$, possibly much bigger than τ, such that $\varepsilon^2/\tilde{\tau} \to 0$ as $\varepsilon \to 0$; let us ask how likely it is to see a near loop of the order of magnitude $\tilde{\tau}$. We claim that for any constant C,

$$\nu(\{\omega : \exists s \exists t \ s \leq r \leq t, |\omega(s) - \omega(t)| \leq \varepsilon, \text{and } C\tilde{\tau} \geq t - s \geq C^{-1}\tilde{\tau}\}) \to 0 \quad (5)$$

as $\varepsilon \to 0$ if $d \geq 4$. This is because Wiener measure is invariant under the scaling transformation $\omega \mapsto \eta$, where $\eta(t) = \alpha^{-1/2}\omega(\alpha t)$ (here for notational convenience we take the time parameter set for the Wiener process to be $[0, \infty)$ rather than $[0, 1]$), so that (5) is equivalent to

$$\nu(\{\omega : \exists s \exists t \ s \leq r \leq t, |\omega(s) - \omega(t)| \leq \varepsilon\tilde{\tau}^{-1/2}, \text{and } C \geq t - s \geq C^{-1}\}) \to 0. \quad (6)$$

But this holds by the continuity with probability one of Wiener paths and the fact that with probability one Wiener paths in dimension $d \geq 4$ have no double points [2]. Nevertheless, and this is what distinguishes the Wiener process in dimension 4 from the Wiener process in dimension $d \geq 5$, it follows from the work of Lawler (see the proof of Theorem 2.6 of [4]) that there is a function τ with $\varepsilon^2/\tau \to 0$ such that in dimension 4 we have

$$\nu(\{\omega : \exists s \exists t \; s \leq r \leq t, |\omega(s) - \omega(t)| \leq \varepsilon, \text{ and } t - s \geq \tau\}) \to 1 \qquad (7)$$

as $\varepsilon \to 0$. Thus in dimension 4 near loops are constantly occurring, but if we try to specify in advance their order of magnitude we do not see any. It is possible that here is a phenomenon that would not be captured by hierarchical models, renormalization group techniques, or computer simulation.

This suggests a possible candidate for a nontrivial polymer measure in dimension 4. For fixed ε, and τ as in (7), let S_ε be the set of all paths that do not have any near loops about any r, and let λ_ε be the restriction of Wiener measure to S_ε, so that $\lambda_\varepsilon(A) = \nu(S_\varepsilon)^{-1} \nu(S_\varepsilon \cap A)$. Since $\nu(S_\varepsilon) \to 0$ by (7), we have picked out a very improbable set of paths. Let λ be any weak limit of λ_ε as $\varepsilon \to 0$. The main question, on which I have made no progress, is this: is a typical sample path of the λ process sufficiently bigger than a Wiener path so that two independent sample paths of the λ process have a nonzero probability of intersecting? The Wiener path itself in dimension 4 just fails, by a logarithmically divergent integral, to have positive capacity, so there are grounds for optimism that the self-repulsion produced by avoiding near loops thickens the path sufficiently to make possible a local interaction in dimension 4.

Note added in proof: This chapter was written before the remarkable results of Aizenman [9] and Fröhlich [10] appeared. These results dampen one's optimism about the speculations made here, but do not eliminate the need for more results about polymer measures in all dimensions.

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