THEORY OF PLATES AND SHELLS

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SECOND EDITION

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PREFACE

Since the publication of the first edition of this book, the application of the theory of plates and shells in practice has widened considerably, and some new methods have been introduced into the theory. To take these facts into consideration, we have had to make many changes and additions. The principal additions are (1) an article on deflection of plates due to transverse shear, (2) an article on stress concentrations around a circular hole in a bent plate, (3) a chapter on bending of plates resting on an elastic foundation, (4) a chapter on bending of anisotropic plates, and (5) a chapter reviewing certain special and approximate methods used in plate analysis. We have also expanded the chapter on large deflections of plates, adding several new cases of plates of variable thickness and some numerical tables facilitating plate analysis.

In the part of the book dealing with the theory of shells, we limited ourselves to the addition of the stress-function method in the membrane theory of shells and some minor additions in the flexural theory of shells.

The theory of shells has been developing rapidly in recent years, and several new books have appeared in this field. Since it was not feasible for us to discuss these new developments in detail, we have merely referred to the new bibliography, in which persons specially interested in this field will find the necessary information.

S. Timoshenko
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NOTATION

x, y, z	Rectangular coordinates
r, θ	Polar coordinates
r_x, r_y	Radii of curvature of the middle surface of a plate in xz and yz planes, respectively
h	Thickness of a plate or a shell
q	Intensity of a continuously distributed load
p	Pressure
P	Single load
γ	Weight per unit volume
$\sigma_x, \sigma_y, \sigma_z$	Normal components of stress parallel to x, y, and z axes
σ_n	Normal component of stress parallel to n direction
σ_r	Radial stress in polar coordinates
σ_t, σ_θ	Tangential stress in polar coordinates
τ	Shearing stress
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	Shearing stress components in rectangular coordinates
u, v, w	Components of displacements
ϵ	Unit elongation
$\epsilon_x, \epsilon_y, \epsilon_z$	Unit elongations in x, y, and z directions
ϵ_r	Radial unit elongation in polar coordinates
$\epsilon_t, \epsilon_\theta$	Tangential unit elongation in polar coordinates
$\epsilon_\varphi, \epsilon_\theta$	Unit elongations of a shell in meridional direction and in the direction of parallel circle, respectively
$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	Shearing strain components in rectangular coordinates
$\gamma_{r\theta}$	Shearing strain in polar coordinates
E	Modulus of elasticity in tension and compression
G	Modulus of elasticity in shear
ν	Poisson's ratio
V	Strain energy
D	Flexural rigidity of a plate or shell
M_x, M_y	Bending moments per unit length of sections of a plate perpendicular to x and y axes, respectively
M_{xy}	Twisting moment per unit length of section of a plate perpendicular to x axis
M_n, M_{nt}	Bending and twisting moments per unit length of a section of a plate perpendicular to n direction
Q_x, Q_y	Shearing forces parallel to z axis per unit length of sections of a plate perpendicular to x and y axes, respectively
Q_n	Shearing force parallel to z axis per unit length of section of a plate perpendicular to n direction
N_x, N_y	Normal forces per unit length of sections of a plate perpendicular to x and y directions, respectively

N_{xy}	Shearing force in direction of y axis per unit length of section of a plate perpendicular to x axis
M_r, M_t, M_{rt}	Radial, tangential, and twisting moments when using polar coordinates
Q_r, Q_t	Radial and tangential shearing forces
N_r, N_t	Normal forces per unit length in radial and tangential directions
r_1, r_2	Radii of curvature of a shell in the form of a surface of revolution in meridional plane and in the normal plane perpendicular to meridian, respectively
χ_φ, χ_θ	Changes of curvature of a shell in meridional plane and in the plane perpendicular to meridian, respectively
$\chi_{\theta\varphi}$	Twist of a shell
X, Y, Z	Components of the intensity of the external load on a shell, parallel to x, y, and z axes, respectively
N_φ, N_θ, $N_{\varphi\theta}$	Membrane forces per unit length of principal normal sections of a shell
M_θ, M_φ	Bending moments in a shell per unit length of meridional section and a section perpendicular to meridian, respectively
χ_x, χ_φ	Changes of curvature of a cylindrical shell in axial plane and in a plane perpendicular to the axis, respectively
N_φ, N_x, $N_{x\varphi}$	Membrane forces per unit length of axial section and a section perpendicular to the axis of a cylindrical shell
M_φ, M_x	Bending moments per unit length of axial section and a section perpendicular to the axis of a cylindrical shell, respectively
$M_{x\varphi}$	Twisting moment per unit length of an axial section of a cylindrical shell
Q_φ, Q_x	Shearing forces parallel to z axis per unit length of an axial section and a section perpendicular to the axis of a cylindrical shell, respectively
log	Natural logarithm
$\log_{10}$, Log	Common logarithm

CONTENTS

INTRODUCTION

The bending properties of a plate depend greatly on its thickness as compared with its other dimensions. In the following discussion, we shall distinguish between three kinds of plates: (1) thin plates with small deflections, (2) thin plates with large deflections, (3) thick plates.

Thin Plates with Small Deflection. If deflections w of a plate are small in comparison with its thickness h, a very satisfactory approximate theory of bending of the plate by lateral loads can be developed by making the following assumptions:

1. There is no deformation in the middle plane of the plate. This plane remains *neutral* during bending.
2. Points of the plate lying initially on a normal-to-the-middle plane of the plate remain on the normal-to-the-middle surface of the plate after bending.
3. The normal stresses in the direction transverse to the plate can be disregarded.

Using these assumptions, all stress components can be expressed by deflection w of the plate, which is a function of the two coordinates in the plane of the plate. This function has to satisfy a linear partial differential equation, which, together with the boundary conditions, completely defines w. Thus the solution of this equation gives all necessary information for calculating stresses at any point of the plate.

The second assumption is equivalent to the disregard of the effect of shear forces on the deflection of plates. This assumption is usually satisfactory, but in some cases (for example, in the case of holes in a plate) the effect of shear becomes important and some corrections in the theory of thin plates should be introduced (see Art. 39).

If, in addition to lateral loads, there are external forces acting in the middle plane of the plate, the first assumption does not hold any more, and it is necessary to take into consideration the effect on bending of the plate of the stresses acting in the middle plane of the plate. This can be done by introducing some additional terms into the above-mentioned differential equation of plates (see Art. 90).

Thin Plates with Large Deflection. The first assumption is completely satisfied only if a plate is bent into a developable surface. In other cases bending of a plate is accompanied by strain in the middle plane, but calculations show that the corresponding stresses in the middle plane are negligible if the deflections of the plate are small in comparison with its thickness. If the deflections are not small, these supplementary stresses must be taken into consideration in deriving the differential equation of plates. In this way we obtain nonlinear equations and the solution of the problem becomes much more complicated (see Art. 96). In the case of large deflections we have also to distinguish between immovable edges and edges free to move in the plane of the plate, which may have a considerable bearing upon the magnitude of deflections and stresses of the plate (see Arts. 99, 100). Owing to the curvature of the deformed middle plane of the plate, the supplementary tensile stresses, which predominate, act in opposition to the given lateral load; thus, the given load is now transmitted partly by the flexural rigidity and partly by a membrane action of the plate. Consequently, very thin plates with negligible resistance to bending behave as membranes, except perhaps for a narrow edge zone where bending may occur because of the boundary conditions imposed on the plate.

The case of a plate bent into a developable, in particular into a cylindrical, surface should be considered as an exception. The deflections of such a plate may be of the order of its thickness without necessarily producing membrane stresses and without affecting the linear character of the theory of bending. Membrane stresses would, however, arise in such a plate if its edges are immovable in its plane and the deflections are sufficiently large (see Art. 2). Therefore, in "plates with small deflection" membrane forces caused by edges immovable in the plane of the plate can be practically disregarded.

Thick Plates. The approximate theories of thin plates, discussed above, become unreliable in the case of plates of considerable thickness, especially in the case of highly concentrated loads. In such a case the thick-plate theory should be applied. This theory considers the problem of plates as a three-dimensional problem of elasticity. The stress analysis becomes, consequently, more involved and, up to now, the problem is completely solved only for a few particular cases. Using this analysis, the necessary corrections to the thin-plate theory at the points of concentrated loads can be introduced.

The main suppositions of the theory of thin plates also form the basis for the usual theory of thin shells. There exists, however, a substantial difference in the behavior of plates and shells under the action of external loading. The static equilibrium of a plate element under a lateral load is only possible by action of bending and twisting moments, usually accompanied by shearing forces, while a shell, in general, is able to transmit the surface load by "membrane" stresses which act parallel to the tangential plane at a given point of the middle surface and are distributed uniformly over the thickness of the shell. This property of shells makes them, as a rule, a much more rigid and a more economical structure than a plate would be under the same conditions.

In principle, the membrane forces are independent of bending and are wholly defined by the conditions of static equilibrium. The methods of determination of these forces represent the so-called "membrane theory of shells." However, the reactive forces and deformation obtained by the use of the membrane theory at the shell's boundary usually become incompatible with the actual boundary conditions. To remove this discrepancy the bending of the shell in the edge zone has to be considered, which may affect slightly the magnitude of initially calculated membrane forces. This bending, however, usually has a very localized[1] character and may be calculated on the basis of the same assumptions which were used in the case of small deflections of thin plates. But there are problems, especially those concerning the elastic stability of shells, in which the assumption of small deflections should be discontinued and the "large-deflection theory" should be used.

If the thickness of a shell is comparable to the radii of curvature, or if we consider stresses near the concentrated forces, a more rigorous theory, similar to the thick-plate theory, should be applied.

[1] There are some kinds of shells, especially those with a negative gaussian curvature, which provide us with a lot of exceptions. In the case of developable surfaces such as cylinders or cones, large deflection without strain of the middle surface is possible, and, in some cases, membrane stresses can be neglected and consideration of the bending stresses alone may be sufficient.

CHAPTER 1

BENDING OF LONG RECTANGULAR PLATES TO A CYLINDRICAL SURFACE

1. Differential Equation for Cylindrical Bending of Plates. We shall begin the theory of bending of plates with the simple problem of the bending of a long rectangular plate that is subjected to a transverse load that does not vary along the length of the plate. The deflected surface of a portion of such a plate at a considerable distance from the ends[1] can be assumed cylindrical, with the axis of the cylinder parallel to the length of the plate. We can therefore restrict ourselves to the investigation of the bending of an elemental strip cut from the plate by two planes perpendicular to the length of the plate and a unit distance (say 1 in.) apart. The deflection of this strip is given by a differential equation which is similar to the deflection equation of a bent beam.

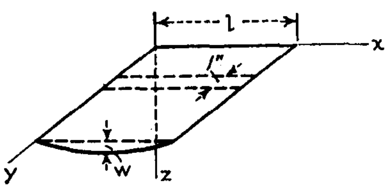

FIG. 1

To obtain the equation for the deflection, we consider a plate of uniform thickness, equal to h, and take the xy plane as the middle plane of the plate before loading, *i.e.*, as the plane midway between the faces of the plate. Let the y axis coincide with one of the longitudinal edges of the plate and let the positive direction of the z axis be downward, as shown in Fig. 1. Then if the width of the plate is denoted by l, the elemental strip may be considered as a bar of rectangular cross section which has a length of l and a depth of h. In calculating the bending stresses in such a bar we assume, as in the ordinary theory of beams, that cross sections of the bar remain plane during bending, so that they undergo only a rotation with respect to their neutral axes. If no normal forces are applied to the end sections of the bar, the neutral surface of the bar coincides with the middle surface of the plate, and the unit elongation of a fiber parallel to the x axis is proportional to its distance z

[1] The relation between the length and the width of a plate in order that the maximum stress may approximate that in an infinitely long plate is discussed later; see pp. 118 and 125.

from the middle surface. The curvature of the deflection curve can be taken equal to $-d^2w/dx^2$, where w, the deflection of the bar in the z direction, is assumed to be small compared with the length of the bar l. The unit elongation ϵ_x of a fiber at a distance z from the middle surface (Fig. 2) is then $-z\, d^2w/dx^2$.

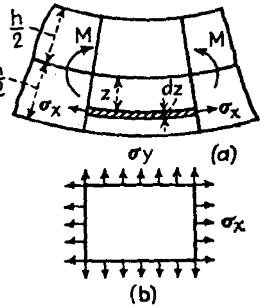

FIG. 2

Making use of Hooke's law, the unit elongations ϵ_x and ϵ_y in terms of the normal stresses σ_x and σ_y acting on the element shown shaded in Fig. 2*a* are

$$\begin{aligned} \epsilon_x &= \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} \\ \epsilon_y &= \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} = 0 \end{aligned} \tag{1}$$

where E is the modulus of elasticity of the material and ν is Poisson's ratio. The lateral strain in the y direction must be zero in order to maintain continuity in the plate during bending, from which it follows by the second of the equations (1) that $\sigma_y = \nu\sigma_x$. Substituting this value in the first of the equations (1), we obtain

$$\epsilon_x = \frac{(1 - \nu^2)\sigma_x}{E}$$

and

$$\sigma_x = \frac{E\epsilon_x}{1 - \nu^2} = -\frac{Ez}{1 - \nu^2}\frac{d^2w}{dx^2} \tag{2}$$

If the plate is submitted to the action of tensile or compressive forces acting in the x direction and uniformly distributed along the longitudinal sides of the plate, the corresponding direct stress must be added to the stress (2) due to bending.

Having the expression for bending stress σ_x, we obtain by integration the bending moment in the elemental strip:

$$M = \int_{-h/2}^{h/2} \sigma_x z\, dz = -\int_{-h/2}^{h/2} \frac{Ez^2}{1 - \nu^2}\frac{d^2w}{dx^2}\, dz = -\frac{Eh^3}{12(1 - \nu^2)}\frac{d^2w}{dx^2}$$

Introducing the notation

$$\frac{Eh^3}{12(1 - \nu^2)} = D \tag{3}$$

we represent the equation for the deflection curve of the elemental strip in the following form:

$$D\frac{d^2w}{dx^2} = -M \tag{4}$$

in which the quantity D, taking the place of the quantity EI in the case of beams, is called the *flexural rigidity* of the plate. It is seen that the calculation of deflections of the plate reduces to the integration of Eq. (4), which has the same form as the differential equation for deflection of beams. If there is only a lateral load acting on the plate and the edges are free to approach each other as deflection occurs, the expression for the bending moment M can be readily derived, and the deflection curve is then obtained by integrating Eq. (4). In practice the problem is more complicated, since the plate is usually attached to the boundary and its edges are not free to move. Such a method of support sets up tensile reactions along the edges as soon as deflection takes place. These reactions depend on the magnitude of the deflection and affect the magnitude of the bending moment M entering in Eq. (4). The problem reduces to the investigation of bending of an elemental strip submitted to the action of a lateral load and also an axial force which depends on the deflection of the strip.[1] In the following we consider this problem for the particular case of uniform load acting on a plate and for various conditions along the edges.

2. Cylindrical Bending of Uniformly Loaded Rectangular Plates with Simply Supported Edges. Let us consider a uniformly loaded long rectangular plate with longitudinal edges which are free to rotate but cannot move toward each other during bending. An elemental strip cut out

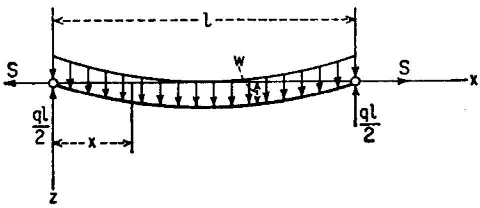

Fig. 3

from this plate, as shown in Fig. 1, is in the condition of a uniformly loaded bar submitted to the action of an axial force S (Fig. 3). The magnitude of S is such as to prevent the ends of the bar from moving along the x axis. Denoting by q the intensity of the uniform load, the bending moment at any cross section of the strip is

$$M = \frac{ql}{2}x - \frac{qx^2}{2} - Sw$$

[1] In such a form the problem was first discussed by I. G. Boobnov; see the English translation of his work in *Trans. Inst. Naval Architects*, vol. 44, p. 15, 1902, and his "Theory of Structure of Ships," vol. 2, p. 545, St. Petersburg, 1914. See also the paper by Stewart Way presented at the National Meeting of Applied Mechanics, ASME, New Haven, Conn., June, 1932; from this paper are taken the curves used in Arts. 2 and 3.