An Introduction to Error Analysis

The Study of Uncertainties in Physical Measurements

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Preface

All measurements, however careful and scientific, are subject to some uncertainties. Error analysis is the study and evaluation of these uncertainties, its two main functions being to allow the scientist to estimate how large his uncertainties are, and to help him to reduce them when necessary. The analysis of uncertainties, or "errors," is a vital part of any scientific experiment, and error analysis is therefore an important part of any college course in experimental science. It can also be one of the most interesting parts of the course. The challenges of estimating uncertainties and of reducing them to a level that allows a proper conclusion to be drawn can turn a dull and routine set of measurements into a truly interesting exercise.

This book is an introduction to error analysis for use with an introductory college course in experimental physics of the sort usually taken by freshmen or sophomores in the sciences or engineering. I certainly do not claim that error analysis is the most (let alone the only) important part of such a course, but I have found that it is often the most abused and neglected part. In many such courses, error analysis is "taught" by handing out a couple of pages of notes containing a few formulas, and the student is then expected to get on with the job solo. The result is that error analysis becomes a meaningless ritual, in which the student adds a few lines of calculation to the end of each laboratory report, not because he or she understands why, but simply because the instructor has said to do so.

I wrote this book with the conviction that any student, even one who has never heard of the subject, should be able to learn what error analysis is, why it is interesting and important, and how to use the basic tools of the subject in laboratory reports. Part I of the book (Chapters 1 to 5) tries to do all this, with many examples of the kind of experiment encountered in teaching laboratories. The student who masters this material should then know and understand almost all the error analysis he or she would be expected to learn in a freshman laboratory course: error propagation, the use of elementary statistics, and their justification in terms of the normal distribution.

Part II contains a selection of more advanced topics: least-squares fitting, the correlation coefficient, the χ^2 test, and others. These would almost certainly not be included officially in a freshman laboratory course, although a few students might become interested in some of them. However, several of these topics would be needed in a second laboratory course, and it is primarily for that reason that I have included them.

I am well aware that there is all too little time to devote to a subject like error analysis in most laboratory courses. At the University of Colorado we give a one-hour lecture in each of the first six weeks of our freshman laboratory course. These lectures, together with a few homework assignments using the problems at the ends of the chapters, have let us cover Chapters 1 through 4 in detail and Chapter 5 briefly. This gives the students a working knowledge of error propagation and the elements of statistics, plus a nodding acquaintance with the underlying theory of the normal distribution.

From several students' comments at Colorado, it was evident that the lectures were an unnecessary luxury for at least some of the students, who could probably have learned the necessary material from assigned reading and problem sets. I certainly believe the book could be studied without any help from lectures.

Part II could be taught in a few lectures at the start of a second-year laboratory course (again supplemented with some assigned problems). But, even more than Part I, it was intended to be read by the student at any time that his or her own needs and interests might dictate. Its seven chapters are almost completely independent of one another, in order to encourage this kind of use.

I have included a selection of problems at the end of each chapter; the reader does need to work several of these to master the techniques. Most calculations of errors are quite straightforward. A student who finds himself or herself doing many complicated calculations (either in the problems of this book or in laboratory reports) is almost certainly doing something in an unnecessarily difficult way. In order to give teachers and readers a good choice, I have included many more problems than the average reader need try. A reader who did one-third of the problems would be doing well.

Inside the front and back covers are summaries of all the principal formulas. I hope the reader will find these a useful reference, both while studying the book and afterward. The summaries are organized by chapters, and will also, I hope, serve as brief reviews to which the reader can turn after studying each chapter.

Within the text, a few statements—equations and rules of procedure—have been highlighted by a shaded background. This highlighting is reserved for statements that are important and are in their final form (that is, will not be modified by later work). You will definitely need to remember these statements; so they have been highlighted to bring them to your attention.

The level of mathematics expected of the reader rises slowly through the book. The first two chapters require only algebra; Chapter 3 requires differentiation (and partial differentiation in Section 3.9, which is optional); Chapter 5 needs a knowledge of integration and the exponential function. In Part II, I assume that the reader is entirely comfortable with all these ideas.

The book contains numerous examples of physics experiments, but an understanding of the underlying theory is not essential. Furthermore, the examples are mostly taken from elementary mechanics and optics, to make it more likely that the student will already have studied the theory. The reader who needs it can find an account of the theory by looking at the index of any introductory physics text.

Error analysis is a subject about which people feel passionately, and no single treatment can hope to please everyone. My own prejudice is that, when a choice has to be made between ease of understanding and strict rigor, a physics text should choose the former. For example, on the controversial question of combining errors in quadrature versus direct addition, I have chosen to treat direct addition first, since the student can easily understand the arguments that lead to it.

In the last few years, a dramatic change has occurred in student laboratories with the advent of the pocket calculator. This has a few unfortunate consequences—most notably, the atrocious habit of quoting ridiculously *insignificant* figures just because the calculator produced them—but it is from almost every point of view a tremendous advantage, especially in error analysis. The pocket calculator allows one to compute, in a few seconds, means and standard deviations that previously would have taken hours. It renders unnecessary many tables, since one can now compute functions like the Gauss function more quickly than one could find them in a book of tables. I have tried to exploit this wonderful tool wherever possible.

It is my pleasure to thank several people for their helpful comments and suggestions. A preliminary edition of the book was used at several colleges, and I am grateful to many students and colleagues for their criticisms. Especially helpful were the comments of John Morrison and David Nesbitt at the University of Colorado, Professors Pratt and Schroeder at Michigan State, Professor Shugart at U. C. Berkeley, and

Professor Semon at Bates College. Diane Casparian, Linda Frueh, and Connie Gurule typed successive drafts beautifully and at great speed. Without my mother-in-law, Frances Kretschmann, the proofreading would never have been done in time. I am grateful to all of these people for their help; but above all I thank my wife, whose painstaking and ruthless editing improved the whole book beyond measure.

J. R. Taylor
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Part I

Part I introduces the basic ideas of error analysis as they are needed in a typical first-year, college physics laboratory. The first two chapters describe what error analysis is, why it is important, and how it can be used in a typical laboratory report. Chapter 3 describes error propagation, whereby uncertainties in one's original measurements "propagate" through calculations to cause uncertainties in one's calculated final answers. Chapters 4 and 5 introduce the statistical methods with which the so-called random uncertainties can be evaluated.

CHAPTER 1

Preliminary Description of Error Analysis

Error analysis is the study and evaluation of uncertainty in measurement. Experience has shown that no measurement, however carefully made, can be completely free of uncertainties. Since the whole structure and application of science depends on measurements, it is therefore crucially important to be able to evaluate these uncertainties and to keep them to a minimum.

In this first chapter we describe some simple measurements that illustrate the inevitable occurrence of experimental uncertainties and show the great importance of knowing how large these uncertainties are. We shall then describe how (in some simple cases, at least) the magnitude of the experimental uncertainties can be realistically estimated, often by means of little more than ordinary common sense.

1.1. Errors as Uncertainties

In science the word "error" does not carry the usual connotations of "mistake" or "blunder." "Error" in a scientific measurement means the inevitable uncertainty that attends all measurements. As such, errors are not mistakes; you cannot avoid them by being very careful. The best you can hope to do is to ensure that errors are as small as reasonably possible, and to have some reliable estimate of how large they are. Most textbooks introduce additional definitions of "error," and we shall discuss some of these later. For the moment, however, we shall use "error" exclusively in the sense of "uncertainty," and treat the two words as being interchangeable.

1.2. Inevitability of Uncertainty

To illustrate the inevitable occurrence of uncertainties, we have only to examine carefully any everyday measurement. Consider, for example, a carpenter who must measure the height of a doorway in order to install a door. As a first rough measurement, he might simply look at the doorway and estimate that it is 210 cm high. This crude "measurement" is certainly subject to uncertainty. If pressed, the carpenter might express this uncertainty by admitting that the height could be as little as 205 or as much as 215 cm.

If he wanted a more accurate measurement, he would use a tape measure, and he might find that the height is 211.3 cm. This measurement is certainly more precise than his original estimate, but it is obviously still subject to some uncertainty, since it is *inconceivable* that he could know the height to be exactly 211.3000 rather than 211.3001 cm, for example.

There are many reasons for this remaining uncertainty, several of which we will be discussing in this book. Some of these causes of uncertainty could be removed if he took enough trouble. For example, one source of uncertainty might be that poor lighting is making it difficult to read the tape; this could be corrected by improving the lighting.

On the other hand, some sources of uncertainty are intrinsic to the process of measurement and can never be entirely removed. For example, let us suppose the carpenter's tape is graduated in half-centimeters. The top of the door will probably not coincide precisely with one of the half-centimeter marks, and if it does not, then the carpenter must *estimate* just where the top lies between two marks. Even if the top happens to coincide with one of the marks, the mark itself is perhaps a millimeter wide; so he must estimate just where the top lies within the mark. In either case, the carpenter ultimately must estimate where the top of the door lies relative to the markings on his tape, and this necessity causes some uncertainty in his answer.

By buying a better tape with closer and finer markings, the carpenter can reduce his uncertainty, but he cannot eliminate it entirely. If he becomes obsessively determined to find the height of the door with the greatest precision that is technically possible, he could buy an expensive laser interferometer. But even the precision of an interferometer is limited to distances of the order of the wavelength of light (about 0.5×10^{-6} meters). Although he would now be able to measure the height with fantastic precision, he still would not know the height of the doorway *exactly*.

Furthermore, as our carpenter strives for greater precision, he will encounter an important problem of principle. He will certainly find that the height is different in different places. Even in one place, he will find that the height varies if the temperature and humidity vary, or even if he accidentally rubs off a thin layer of dirt. In other words, he will find that there is no such thing as *the* height of the doorway. This kind of problem is called a *problem of definition* (the height of the door is not a well-defined quantity) and plays an important role in many scientific measurements.

Our carpenter's experiences illustrate what is found to be generally true. No physical quantity (a length, a time, a temperature, etc.) can be measured with complete certainty. With care we may be able to reduce the uncertainties until they are extremely small, but to eliminate them entirely is impossible.

In everyday measurements we do not usually bother to discuss uncertainties. Sometimes the uncertainties simply are not interesting. If we say that the distance between home and school is 3 miles, it does not matter (for most purposes) whether this means "somewhere between 2.5 and 3.5 miles" or "somewhere between 2.99 and 3.01 miles." Often the uncertainties are important, but can be allowed for instinctively and without explicit consideration. When our carpenter comes to fit his door, he must know its height with an uncertainty that is less than 1 mm or so. However, as long as the uncertainty is this small, the door will (for all practical purposes) be a perfect fit, and his concern with error analysis is at an end.

1.3. Importance of Knowing the Uncertainties

Our example of the carpenter measuring a doorway illustrated how there are always uncertainties in measurements. We will now consider an example that illustrates more clearly the crucial importance of knowing how big these uncertainties are.

Suppose we are faced with a problem like the one said to have been solved by Archimedes. We are asked to find our whether a crown is made of 18-karat gold, as claimed, or is a cheaper alloy. Following Archimedes, we decide to test the crown's density, knowing that the densities of 18-karat gold and the suspected alloy are

$$\rho_{\text{gold}} = 15.5 \text{ gm/cm}^3$$

and

$$\rho_{\text{alloy}} = 13.8 \text{ gm/cm}^3.$$

If we can measure the density ρ_{crown} of the crown, then it should be possible (as Archimedes suggested) for us to decide whether the crown is really gold, by comparing ρ_{crown} with the known densities ρ_{gold} and ρ_{alloy}.

Suppose we summon two experts in the measurement of density. The first expert, A, might make a quick measurement of ρ_{crown} and report that his best estimate for ρ_{crown} is 15, and that ρ_{crown} almost certainly lies somewhere between 13.5 and 16.5 gm/cm^3. Expert *B* might take a little longer, and then report a best estimate of 13.9 and a probable range from 13.7 to 14.1 gm/cm^3. The findings of our two experts can be summarized as shown in Table 1.1.

Table 1.1. Density of crown (in gm/cm^3).

Measurement reported	Expert *A*	Expert *B*
Best estimate for ρ_{crown}	15	13.9
Probable range for ρ_{crown}	13.5 to 16.5	13.7 to 14.1

The first point to notice about these results is that although *B*'s measurement is much more precise, *A*'s measurement is probably correct also. Each expert states a range within which he is confident ρ_{crown} lies, and these ranges overlap; so it is perfectly possible (and in fact probable) that both statements are correct.

The next point to notice is that the uncertainty in *A*'s measurement is so large that his results are of no use. The densities of 18-karat gold and of the alloy both lie in his range, 13.5 to 16.5 gm/cm^3; so it is impossible to draw any conclusion from *A*'s measurements. On the other hand, *B*'s measurements indicate clearly that the crown is not genuine; the density of the suspected alloy, 13.8, lies comfortably inside *B*'s estimated range of 13.7 to 14.1, but that of 18-karat gold, 15.5, is well outside it. Evidently, if the measurements are to allow a conclusion, the experimental uncertainties must not be too large. However, it is *not* necessary that the uncertainties be extremely small. In this respect our example is typical of many scientific measurements, where uncertainties have to be reasonably small (perhaps a few percent of the measured value), but where extreme precision is often quite unnecessary.

Since our decision hinges on *B*'s claim that ρ_{crown} lies between 13.7 and 14.1 gm/cm^3, it is important that *B* give us sufficient reason to believe his claim. In other words, the experimenter must justify his stated range of values. This point is often overlooked by the beginning student, who simply asserts that his uncertainty was 1 mm, or 2 sec, or whatever, omitting any justifications. Without a brief explanation of how the uncertainty was estimated, the assertion is almost useless.

The most important point about our two experts' measurements is this: like most scientific measurements, they would both have been useless, if they had not included reliable statements of their uncertainties. In fact, if we knew only the information on the top line of Table 1.1, not only