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Solitons in Field Theory and Nonlinear Analysis

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Preface

There are many interesting and challenging problems in the area of classical field theory. This area has attracted the attention of algebraists, geometers, and topologists in the past and has begun to attract more analysts. Analytically, classical field theory offers all types of differential equation problems which come from the two basic sets of equations in physics describing fundamental interactions, namely, the Yang–Mills equations governing electromagnetic, weak, and strong forces, reflecting internal symmetry, and the Einstein equations governing gravity, reflecting external symmetry. Naturally, a combination of these two sets of equations would lead to a theory which couples both symmetries and unifies all forces, at the classical level. This book is a monograph on the analysis and solution of the nonlinear static equations arising in classical field theory.

It is well known that many important physical phenomena are the consequences of various levels of symmetry breakings, internal or external, or both. These phenomena are manifested through the presence of locally concentrated solutions of the corresponding governing equations, giving rise to physical entities such as electric point charges, gravitational blackholes, cosmic strings, superconducting vortices, monopoles, dyons, and instantons. The study of these types of solutions, commonly referred to as solitons due to their particle-like behavior in interactions, except blackholes, is the subject of this book.

There are two approaches in the study of differential equations of field theory. The first one is to find closed-form solutions. Such an approach works only for a narrow category of problems known as integrable equations, and, in each individual case, the solution often depends heavily on an ingenious construction. The second one, which will be the main focus of this book, is to investigate the solutions using tools from modern nonlinear analysis, an approach initiated by A. Jaffe and C. H. Taubes in their study of the Ginzburg–Landau vortices and Yang–Mills monopoles (*Vortices and Monopoles*, Birkhäuser, 1980).

The book is divided into 12 chapters. In Chapter 1, we present a short introduction to classical field theory, emphasizing the basic concepts and terminology that will be encountered in subsequent chapters. In Chapters 2–12, we present the subject work of the book, namely, solitons as locally concentrated static solutions of field equations and nonlinear functional analysis. In the last section of each of these chapters, we propose some open problems.

The main purpose of Chapter 1 is to provide a quick (in 40 or so pages) and self-contained mathematical introduction to classical field theory. We start from the canonical description of the Newtonian mechanics and the motion of a charged particle in an electromagnetic field. As a consequence, we will see the natural need of a gauge field when quantum mechanical motion is considered via the Schrödinger equation. We then present special relativity and its action principle formulation, which gives birth to the Born–Infeld theory, as will be seen in Chapter 12. We also use special relativity to derive the Klein–Gordon wave equations and the Maxwell equations. After this, we study the important role of symmetry and prove Noether's theorem. In particular, we shall see the origins of some important physical quantities such as energy, momentum, charges, and currents. We next present gauge field theory, in particular, the Yang–Mills theory, as a consequence of maintaining local internal symmetry. Related notions, such as symmetry-breaking, the Goldstone particles, and the Higgs mechanism, will be discussed. Finally, we derive the Einstein equations of general relativity and their simplest gravitational implications. In particular, we explain the origins of the metric energy-momentum tensor and the cosmological constant.

In Chapter 2, we start our study of field equations from the 'most integrable' problem: the nonlinear sigma model and its extension by B. J. Schroers containing a gauge field. We first review the elegant explicit solution by A. A. Belavin and A. M. Polyakov of the classical sigma model. We then present the gauged sigma model of Schroers and state what we know about it. The interesting thing is that, although the solutions are topological and stratified energetically as the Belavin–Polyakov solutions, their magnetic fluxes are continuous. We shall see that the governing equation of the gauged sigma model cannot be integrated explicitly and a rigorous understanding of it requires nonlinear analysis based on the weighted Sobolev spaces.

In Chapter 3, we present an existence theory for the self-dual Yang–Mills instantons in all $4m$ Euclidean dimensions. The celebrated Hodge theorem states that, on a compact oriented manifold, each de Rham cohomology class can be represented by a harmonic form. In the Yang–Mills theory, there is a beautiful parallel statement: each second Chern–Pontryagin class on S^4 can be represented by a family of self-dual or anti-self-dual instantons. The purpose of this chapter is to obtain a general representation theorem in all S^{4m}, $m = 1, 2, \cdots$, settings. We first review the unit charge instantons in 4 dimensions by G. 't Hooft (and A. M. Polyakov). As a preparation for E. Witten's charge N solutions, we present the Liouville equation and its explicit solution. We then introduce Witten's solution in 4 dimensions, which motivates our general approach in all $4m$ dimensions. We next review the $4m$-dimensional Yang–Mills theory of D. H. Tchrakian (the 8-dimensional case was also due to B. Grossman, T. W. Kephart, and J. D. Stasheff) and use a dimensional reduction technique to arrive at a system of 2-dimensional equations generalizing Witten's equations. This system will further be reduced into a quasilinear elliptic equation over the Poincaré half-plane and solved using the calculus of variations and a limiting argument.

In Chapter 4, we introduce the generalized Abelian Higgs equations, governing an arbitrary number of complex Higgs fields through electromagnetic interactions. These equations are discovered by B. J. Schroers in his study of linear sigma models and contain as special cases the equations recently found in the electroweak theory with double Higgs fields by G. Bimonte and G. Lozano and a supersymmetric electroweak theory by J. D. Edelstein and C. Nunez. Using the Cholesky decomposition theorem, we shall obtain a complete understanding of these equations defined either on a closed surface or the full plane. When the vacuum symmetry is partially broken, we give some nonexistence results.

In Chapter 5, we start our study of the Chern–Simons equations from the Abelian case. The Chern–Simons theory generally refers to a wide category of field-theoretical models in one temporal and two spatial dimensions that contain a Chern–Simons term in their action densities. These models are relevant in several important problems in condensed matter physics such as high-temperature superconductivity and quantum and fractional Hall effect. In their full generality, the Chern–Simons models are very difficult to analyze and only numerical simulations are possible. However, since the seminal work of J. Hong, Y. Kim, and P.-Y. Pac and R. Jackiw and E. J. Weinberg on the discovery of the self-dual Abelian Chern–Simons equations, considerable progress has been made on the solutions of various simplified models along the line of these self-dual equations, Abelian and non-Abelian, non-relativistic and relativistic. This chapter presents a complete picture of our rigorous understanding of the Abelian self-dual equations: topological and nontopological solutions, quantized and continuous charges and fluxes, existence, nonexistence, and degeneracy (nonuniqueness) of spatially periodic solutions.

In Chapter 6, we study the non-Abelian Chern–Simons equations. In order to study these equations, we need a minimum grasp of the classification theory of the Lie algebras. Thus we first present a self-contained review on some basic notions such as the Cartan–Weyl bases, root vectors, and Cartan matrices. We next consider the self-dual reduction of G. Dunne, R. Jackiw, S.-Y. Pi, and C. Trugenberger for the non-Abelian gauged Schrödinger equations for which the gauge fields obey a Chern–Simons dynamics and the coupled system is non-relativistic. We show how this system may be reduced into a Toda system, with a Cartan matrix as its coefficient matrix. We then present the solution of the Toda system due to A. N. Leznov in the case that the gauge group is $SU(N)$ and write down the explicit solution for the original non-relativistic Chern–Simons equations. After this we begin our study of the non-Abelian relativistic Chern–Simons equations. We shall prove the existence of topological solutions for a more general nonlinear elliptic system for which the coefficient matrix is not necessarily a Cartan matrix. We shall also discuss several illustrative examples.

In Chapter 7, we present a series of existence theorems for electroweak vortices. It is well known that the electroweak theory does not allow vortex-like solutions in the usual sense due to the vacuum structure of the theory. More precisely, vortices in the Abelian Higgs or the Ginzburg–Landau theory occur at the zeros of the Higgs field as topological defects and are thus viewed as the Higgs particle condensed vortices but there can be no finite-energy Higgs particle condensed vortex solutions in the electroweak theory. However, J. Ambjorn and P. Olesen found in their joint work that spatially periodic electroweak vortices occur as a result of the W-particle condensation. This problem has many new features, both physical and mathematical. We shall first present our solution to a simplified system describing the interaction of the W-particles with the weak gauge field. We then introduce the work of Ambjorn–Olesen on the W-condensed vortex equations arising from the classical Weinberg–Salam electroweak theory and state our existence theorem. The Campbell–Hausdorff formula will be a crucial tool in the proof that the spatial periodicity conditions under the original non-Abelian gauge group and under the Abelian gauge group in the unitary gauge are equivalent. Our mathematical analysis of the problem will be based on a multiply constrained variational principle. Finally we present a complete existence theory for the multivortex equations discovered by G. Bimonte and G. Lozano in their study of the two-Higgs electroweak theory.

In Chapter 8, we present existence theorems for electrically and magnetically charged static solutions, known as dyons, in the Georgi–Glashow theory and in the Weinberg–Salam theory. We first review the fundamental idea of P. A. M. Dirac on electromagnetic duality and the existence of a magnetic monopole in the Maxwell theory. We will not elaborate on the original derivation of the charge quantization condition of Dirac based on considering the quantum-mechanical motion of an electric charge in the field of a magnetic monopole but will use directly the fiber bundle devise due to T. T. Wu and C. N. Yang to arrive at the same conclusion. We then present the argument of J. Schwinger for the existence of dyons in the

Maxwell theory and state Schwinger's extended charge quantization formula. We next introduce the work of B. Julia and A. Zee on the existence of dyons in the simplest non-Abelian gauge field theory, the Georgi–Glashow theory. The physical significance of such solutions is that, unlike the Dirac monopoles and Schwinger dyons, the Julia–Zee dyons carry finite energies. We will first present the explicit dyon solutions due to E. B. Bogomol'nyi, M. K. Prasad, and C. M. Sommerfield known as the BPS solutions. Away from the BPS limit, the equations cannot be solved explicitly. In fact, the existence of electricity leads us to a complicated system of nonlinear equations that can only be solved through finding critical points of an indefinite action functional. Recently, Y. M. Cho and D. Maison suggested that dyons, of infinite energy like the Dirac monopoles, exist in the Weinberg–Salam theory. Mathematically, the existence problem of these Weinberg–Salam or Cho–Maison dyons is the same as that of the Julia–Zee dyons in non-BPS limit: the solution depends on the optimization of an indefinite action functional and requires new techniques. In this chapter, we show how to solve these problems involving indefinite functionals. These techniques will have powerful applications to other problems of similar structure.

In Chapter 9, we concentrate on the radially symmetric solutions of a nonlinear scalar equation with a single Dirac source term. We shall use a dynamical system approach to study the reduced ordinary differential equation. The results obtained for this equation may be used to achieve a profound understanding of many field equation problems of the same nonlinearity. For example, for the Abelian Chern–Simons equation, we will use the results to prove that the radially symmetric topological solution is unique and the charges of nontopological solutions fill up an explicitly determined open interval, of any given vortex number; for the cosmic string problem, we will derive a necessary and sufficient condition for the existence of symmetric finite-energy N-string solutions over $\mathbb{R}^2$ and S^2.

In Chapter 10, we study cosmic strings as static solutions of the coupled Einstein and Yang–Mills field equations. It is well accepted that the universe has undergone a series of phase transitions characterized by a sequence of spontaneous symmetry-breakings which can be described by quantum field theory models of various gauge groups. Cosmic strings appear as mixed states due to a broken symmetry which give rise to a multi-centered display of energy and curvature and may serve as seeds for matter accretion for galaxy formation in the early universe, as described in the work of T. W. B. Kibble and A. Vilenkin. Since the problem involves the Einstein equations, a rigorous mathematical construction of such solutions in general is extremely hard, or in fact, impossible. In their independent studies, B. Linet, and A. Comtet and G. W. Gibbons found that the coupled Einstein and Abelian Higgs equations allow a self-dual reduction as in the case of the Abelian Higgs theory without gravity and they pointed out that one might obtain multi-centered string solutions along the line of the work of Jaffe–Taubes. In the main body of this chapter, we present a fairly complete understanding of these multi-centered cosmic string solutions. In particular, we show that there are striking new surprises due to the presence of gravity. For example, we prove that the inverse of Newton's gravitational constant places an explicit upper limit for the total string number. In the later part of this chapter, we combine the ideas of Linet, Comtet–Gibbons, and Ambjorn Olesen to investigate the existence of multi-centered, electroweak, cosmic strings in the coupled Einstein and Weinberg–Salam equations. We shall see that consistency requires a uniquely determined positive cosmological constant. We will begin this chapter with a brief discussion of some basic notions such as string-induced energy and curvature concentration, deficit angle, and conical geometry.

In Chapter 11, we consider a field theory that allows the coexistence of static vortices and anti-vortices, or strings and anti-strings, of opposite magnetic behavior, both local and global. This theory originates from the gauged sigma model of B. J. Schroers with a broken symmetry and has numerous interesting properties. The magnetic fluxes generated from opposite vortices or strings annihilate each other but the energies simply add up as do so for particles. Gravitationally, strings and anti-strings make identical contributions to the total curvature and are equally responsible for the geodesic completeness of the induced metric. Hence, vortices and anti-vortices, or strings and anti-strings, are indistinguishable and there is a perfect symmetry between them. However, the presence of a weak external field can break such a symmetry which triggers the dominance of one of the two types of vortices or strings. Mathematically, this theory introduces a new topological invariant in field theory, the Thom class. A by-product is that these vortices and anti-vortices may be used to construct maps with all possible half-integer 'degrees' defined as topological integrals. As in the Abelian Higgs theory case, the existence of such strings and anti-strings implies a vanishing cosmological constant.

In Chapter 12, we study the solutions of the geometric (nonlinear) theory of electromagnetism of M. Born and L. Infeld which was introduced to accommodate a finite-energy point electric charge modelling the electron and has become one of the major focuses of recent research activities of field theoreticians due to its relevance in superstrings and supermembranes. Mathematically, the Born–Infeld theory is closely related to the minimal surface type problems and presents new opportunities and structure for analysts. We begin this chapter with a short introduction to the Born–Infeld theory and show how the theory allows the existence of finite-energy point charges, electrical or magnetical. We then discuss the electrostatic and magnetostatic problems and relate them to the minimal surface equations and the Bernstein theorems. We shall also obtain a generalized Bernstein problem expressed in terms of differential forms. We next study the Born–Infeld wave equations and show that there is no more Derrick's theorem type constraint on the spatial dimensions for the static problem. Finally we obtain multiple strings or vortices for the Born–Infeld theory coupled with a Higgs field, originally proposed in the work of K. Shiraishi and S. Hirenzaki. In particular, we show that the Born–Infeld parameter plays an important role for the behavior of solutions, both locally and globally.

Bibliographical notes on the development in other topic areas will also be made at appropriate places in the book.

In developing the subjects presented in this work, I have benefited from helpful communications, conversations, and in several cases collaborations with many mathematicians and physicists: S. Adler, J. Ambjorn, H. Brezis, L. A. Caffarelli, X. Chen, Y. M. Cho, G. Dunne, Weinan E, A. Friedman, Y. Z. Guo, S. Hastings, D. Hoffman, R. Jackiw, H. T. Ku, G. P. Li, F. H. Lin, J. B. Mcleod, E. Miller, P. Olesen, B. J. Schroers, L. M. Sibner, R. J. Sibner, T. Spencer, J. Spruck, G. Tarantello, D. H. Tchrakian, E. J. Weinberg, E. Witten, D. Yang. In particular, I wish to thank J. Spruck for initiating my interest in this area and for some fruitful joint work. Finally, I am grateful to my parents, Zhaoqi Yang and Hua Han, and my brothers, Wei Yang and Jin Yang, for their unwavering encouragement and support over the years.

I hope that this book will be useful to both mathematicians and theoretical physicists, especially those interested in nonlinear analysis and its applications.

Brooklyn, New York
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Yisong Yang

Notation and Convention

The signature of an $(n+1)$-dimensional Minkowski spacetime is always $(+-\cdots-)$. The $(n+1)$-dimensional flat Minkowski spacetime is denoted by $\mathbb{R}^{n,1}$ and is equipped with the inner product

$$xy = x^0y^0 - x^1y^1 - \cdots - x^ny^n,$$

where $x=(x^0,x^1,\cdots,x^n), y=(y^0,y^1,\cdots,y^n)\in\mathbb{R}^{n,1}$ are spacetime vectors.

Unless otherwise stated, we always use the Greek letters α,β,μ,ν to denote the spacetime indices,

$$\alpha,\beta,\mu,\nu = 0,1,2,\cdots,n,$$

and the Latin letters i,j,k to denote the space indices,

$$i,j,k = 1,2,\cdots,n.$$

The standard summation convention over repeated indices will be observed. For example,

$$a_ib_i = \sum_{i=1}^n a_ib_i,\quad a_ib^i = \sum_{i=1}^n a_ib^i,\quad a_i^2 = \sum_{i=1}^n a_i^2,\quad |b^i|^2 = \sum_{i=1}^n |b^i|^2.$$

Similarly,

$$F_{jk}^2 = \sum_{j,k=1}^n F_{jk}^2.$$

The roman letter e is reserved to denote the base of the natural logarithmic system and the italic letter e is reserved to denote an irrelevant physical coupling constant such as the charge of a positron ($-e$ will then be the charge of an electron), except within some special statement environment such as a theorem where italic type is used throughout, which should not cause confusion. Likewise, the roman letter i denotes the imaginary unit $\sqrt{-1}$ and the italic letter i is an integer-valued index. We use the roman type abbreviation supp to denote the support of a function. However, within some special statement environment such as a theorem, this abbreviation will also be printed in italic type.

The letter C will be used to denote a positive constant which may assume different values at different places.

For a complex number c, we use $\bar{c}$ to denote its conjugate. For a complex matrix A, we use $A^\dagger$ to denote its Hermitian conjugate, which consists of a matrix transposition and a complex conjugation.

When a system consists of several equations or relations, we often number the system by labelling its last equation or relation.

Although n denotes an integer, the symbol $\partial/\partial n$ stands for the outward normal differentiation on the boundary of a domain.

The symbol $W^{k,p}$ denotes the Sobolev space of functions whose distributional derivatives up to the kth order are all in the space L^p.

By convention, various matrix Lie algebras are denoted by lowercase letters. For example, the Lie algebras of the Lie groups $SO(N)$ and $SU(N)$ are denoted by $so(N)$ and $su(N)$, respectively.

The notation for various derivatives is as follows,

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \partial_\pm = \partial_1 \pm \mathrm{i}\partial_2, \quad \partial = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2), \quad \overline{\partial} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2).$$

Besides, with the complex variable $z = x^1 + \mathrm{i}x^2$, we always understand that $\partial_z = \partial/\partial z = \partial, \partial_{\overline{z}} = \partial/\partial\overline{z} = \overline{\partial}$. Thus, for any function f that only has partial derivatives with respect to x^1 and x^2, the quantities $\partial_z f = \partial f/\partial z$ and $\partial_{\overline{z}} = \partial f/\partial \overline{z}$ are well defined.

Vectors and tensors are often simply denoted by their general components, respectively, following physics literature. For example, it is undertood that

$$A_\mu \equiv (A_\mu) = (A_0, A_1, A_2, A_3), \quad g_{\mu\nu} \equiv (g_{\mu\nu}).$$

In a volume of this scope, it is inevitable to have a letter carry different but standard meanings in different contexts, although such a multiple usage of letters has kept to a minimum. Here are some examples: r may stand for the radial variable or the rank of a Lie group; δ may stand for a small positive number or variation of a functional, δ_p stands for the Dirac distribution concentrated at the point p, and δ_{ij} is the Kronecker symbol; g may stand for a coupling constant, a metric tensor or its determinant, or a function.

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