

***Probability and Its Applications***

**Mu-Fa Chen**

**Eigenvalues,  
Inequalities, and  
Ergodic Theory**



**Springer**

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# **Eigenvalues, Inequalities, and Ergodic Theory**

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# Probability and Its Applications

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# Probability and Its Applications

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# Preface

First, let us explain the precise meaning of the compressed title. The word “eigenvalues” means the first nontrivial Neumann or Dirichlet eigenvalues, or the principal eigenvalues. The word “inequalities” means the Poincaré inequalities, the logarithmic Sobolev inequalities, the Nash inequalities, and so on. Actually, the first eigenvalues can be described by some Poincaré inequalities, and so the second topic has a wider range than the first one. Next, for a Markov process, corresponding to its operator, each inequality describes a type of ergodicity. Thus, study of the inequalities and their relations provides a way to develop the ergodic theory for Markov processes. Due to these facts, from a probabilistic point of view, the book can also be regarded as a study of “ergodic convergence rates of Markov processes,” which could serve as an alternative title of the book. However, this book is aimed at a larger class of readers, not only probabilists.

The importance of these topics should be obvious. On the one hand, the first eigenvalue is the leading term in the spectrum, which plays an important role in almost every branch of mathematics. On the other hand, the ergodic convergence rates constitute a recent research area in the theory of Markov processes. This study has a very wide range of applications. In particular, it provides a tool to describe the phase transitions and the effectiveness of random algorithms, which are now a very fashionable research area.

This book surveys, in a popular way, the main progress made in the field by our group. It consists of ten chapters plus two appendixes. The first chapter is an overview of the second to the eighth ones. Mainly, we study several different inequalities or different types of convergence by using three mathematical tools: a probabilistic tool, the coupling methods (Chapters 2 and 3); a generalized Cheeger’s method originating in Riemannian geometry (Chapter 4); and an approach coming from potential theory and harmonic analysis (Chapters 6 and 7). The explicit criteria for different types of convergence and the explicit estimates of the convergence rates (or the optimal constants in the inequalities) in dimension one are given in Chapters 5 and 6; some generalizations are given in Chapter 7. The proofs of a diagram of nine types of ergodicity (Theorem 1.9) are presented in Chapter 8. Very often, we deal with one-dimensional elliptic operators or tridiagonal matrices (which can be infinite) in detail, but we also handle general differential and integral oper-

ators. To avoid heavy technical details, some proofs are split among several locations in the text. This also provides different views of the same problem at different levels. The topics of the last two chapters (9 and 10) are different but closely related. Chapter 9 surveys the study of a class of interacting particle systems (from which a large part of the problems studied in this book are motivated), and illustrates some applications. In the last chapter, one can see an interesting application of the first eigenvalue, its eigenfunctions, and an ergodic theorem to stochastic models of economics. Some related open problems are included in each chapter. Moreover, an effort is made to make each chapter, except the first one, more or less self-contained. Thus, once one has read about the program in Chapter 1, one may freely go on to the other chapters. The main exception is Chapter 3, which depends heavily on Chapter 2. As usual, a quick way to get an impression about what is done in the book is to look at the summaries given at the beginning of each chapter.

One should not be disappointed if one cannot find an answer in the book for one's own model. The complete solutions to our problems have only recently been obtained in dimension one. Nevertheless, it is hoped that the three methods studied in the book will be helpful. Each method has its own advantages and disadvantages. In principle, the coupling method can produce sharper estimates than the other two methods, but additional work is required to figure out a suitable coupling and, more seriously, a good distance. The Cheeger and capacity methods work in a very general setup and are powerful qualitatively, but they leave the estimation of isoperimetric constants to the reader. The last task is usually quite hard in higher-dimensional situations.

This book serves as an introduction to a developing field. We emphasize the ideas through simple examples rather than technical proofs, and most of them are only sketched. It is hoped that the book will be readable by nonspecialists. In the past ten years or more, the author has tried rather hard to make acceptable lectures; the present book is based on these lecture notes: Chen (1994b; 1997a; 1998a; 1999c; 2001a; 2002b; 2002c; 2003b; 2004a; 2004b) [see Chen (2001c)]. Having presented eleven lectures in Japan in 2002, the author understood that it would be worthwhile to publish a short book, and then the job was started.

Since each topic discussed in the book has a long history and contains a great number of publications, it is impossible to collect a complete list of references. We emphasize the recent progress and related references. It is hoped that the bibliography is still rich enough that the reader can discover a large number of contributors in the field and more related references.

# Acknowledgments

As mentioned before, this book is based on lecture notes presented over the past ten years or so. Thus, the book should be dedicated, with the author's deep acknowledgment, to the mathematicians and their universities/institutes whose kind invitations, financial support, and warm hospitality made those lectures possible. Without their encouragement and effort, the book would never exist. With the kind permission of his readers, the author is happy to list some of the names below (since 1993), with an apology to those that are missing:

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- International Congress of Mathematicians (August 2002, Beijing).
- The First Sino-German Conference on Stochastic Analysis—A Satellite Conference of ICM 2002 (September 2002, Beijing).
- Stochastic Analysis in Infinite Dimensional Spaces (November 2002, Kyoto)
- Japanese National Conference on Stochastic Processes and Related Fields (December 2002, Tokyo).

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# Chapter 1

## An Overview of the Book

This chapter is an overview of the book, especially of the first eight chapters. It consists of four sections. In the first section, we explain what eigenvalues we are interested in and show the difficulties in studying the first (nontrivial) eigenvalue through elementary examples. The second section presents some new (dual) variational formulas and explicit bounds for the first eigenvalue of the Laplacian on Riemannian manifolds or Jacobi matrices (Markov chains), and explains the main idea of the proof, which is a probabilistic approach: the coupling methods. In the third section, we introduce some recent lower bounds of several basic inequalities, based on a generalization of Cheeger's approach which comes from Riemannian geometry. In the last section, a diagram of nine different types of ergodicity and a table of explicit criteria for them are presented. The criteria are motivated by the weighted Hardy inequality, which comes from harmonic analysis.

### 1.1 Introduction

Let me now explain what eigenvalue we are talking about.

#### **Definition. The first (nontrivial) eigenvalue**

Consider a tridiagonal matrix (or in probabilistic language, a birth-death process with state space  $E = \{0, 1, 2, \dots\}$  and  $Q$ -matrix)

$$Q = (q_{ij}) = \begin{pmatrix} -b_0 & b_0 & 0 & 0 & \dots \\ a_1 & -(a_1 + b_1) & b_1 & 0 & \dots \\ 0 & a_2 & -(a_2 + b_2) & b_2 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where  $a_k, b_k > 0$ . Since the sum of each row equals 0, we have  $Q\mathbf{1} = \mathbf{0} = 0 \cdot \mathbf{1}$ , where  $\mathbf{1}$  is the vector having elements 1 everywhere and  $\mathbf{0}$  is the zero vector.

This means that the  $Q$ -matrix has an eigenvalue 0 with eigenvector  $\mathbf{1}$ . Next, consider the finite case  $E_n = \{0, 1, \dots, n\}$ . Then, the eigenvalues of  $-Q$  are discrete:  $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_n$ . We are interested in the first (nontrivial) eigenvalue  $\lambda_1 = \lambda_1 - \lambda_0 =: \text{gap}(Q)$  (also called the *spectral gap* of  $Q$ ). In the infinite case,  $\lambda_1 := \inf\{\text{Spectrum of } (-Q)\} \setminus \{0\}$  can be 0. Certainly, one can consider a self-adjoint elliptic operator in  $\mathbb{R}^d$  or the Laplacian  $\Delta$  on manifolds or an infinite-dimensional operator as in the study of interacting particle systems.

Since the spectral theory is of central importance in many branches of mathematics and the first nontrivial eigenvalue is the leading term of the spectrum, it should not be surprising that the study of  $\lambda_1$  has a very wide range of applications.

## Difficulties

To get a concrete feeling about the difficulties of the topic, let us look at the following examples with finite state spaces.

When  $E = \{0, 1\}$ , it is trivial that  $\lambda_1 = a_1 + b_0$ . Everyone is happy to see this result, since if either  $a_1$  or  $b_0$  increases, so does  $\lambda_1$ . If we go one more step,  $E = \{0, 1, 2\}$ , then we have four parameters,  $b_0, b_1$  and  $a_1, a_2$ . In this case,  $\lambda_1 = 2^{-1}[a_1 + a_2 + b_0 + b_1 - \sqrt{(a_1 - a_2 + b_0 - b_1)^2 + 4a_1b_1}]$ . It is disappointing to see this result, since parameters effect on  $\lambda_1$  is not clear at all. When  $E = \{0, 1, 2, 3\}$ , we have six parameters:  $b_0, b_1, b_2, a_1, a_2, a_3$ . The solution is expressed by the three quantities  $B, C$ , and  $D$ :

$$\lambda_1 = \frac{D}{3} - \frac{C}{3 \cdot 2^{1/3}} + \frac{2^{1/3} (3B - D^2)}{3C},$$

where the quantities  $D, B$ , and  $C$  are not too complicated:

$$D = a_1 + a_2 + a_3 + b_0 + b_1 + b_2,$$

$$B = a_3 b_0 + a_2 (a_3 + b_0) + a_3 b_1 + b_0 b_1 + b_0 b_2 + b_1 b_2 + a_1 (a_2 + a_3 + b_2),$$

$$C = \left( A + \sqrt{4(3B - D^2)^3 + A^2} \right)^{1/3}.$$

However, in the last expression, another quantity,  $A$ , is involved. What, then, is  $A$ ?

$$\begin{aligned} A = & -2a_1^3 - 2a_2^3 - 2a_3^3 + 3a_3^2b_0 + 3a_3b_0^2 - 2b_0^3 + 3a_3^2b_1 - 12a_3b_0b_1 + 3b_0^2b_1 \\ & + 3a_3b_1^2 + 3b_0b_1^2 - 2b_1^3 - 6a_3^2b_2 + 6a_3b_0b_2 + 3b_0^2b_2 + 6a_3b_1b_2 - 12b_0b_1b_2 \\ & + 3b_1^2b_2 - 6a_3b_2^2 + 3b_0b_2^2 + 3b_1b_2^2 - 2b_2^3 + 3a_1^2(a_2 + a_3 - 2b_0 - 2b_1 + b_2) \\ & + 3a_2^2[a_3 + b_0 - 2(b_1 + b_2)] \\ & + 3a_2[a_3^2 + b_0^2 - 2b_1^2 - b_1b_2 - 2b_2^2 - a_3(4b_0 - 2b_1 + b_2) + 2b_0(b_1 + b_2)] \\ & + 3a_1[a_2^2 + a_3^2 - 2b_0^2 - b_0b_1 - 2b_1^2 - a_2(4a_3 - 2b_0 + b_1 - 2b_2) \\ & + 2b_0b_2 + 2b_1b_2 + b_2^2 + 2a_3(b_0 + b_1 + b_2)], \end{aligned}$$

computed using Mathematica. One should be shocked, at least I was, to see this result, since the roles of the parameters are completely hidden! Of course, everyone understands that it is impossible to compute  $\lambda_1$  explicitly when the size of the matrix is greater than five!

Now, how about the estimation of  $\lambda_1$ ? To see this, let us consider the perturbation of the eigenvalues and eigenfunctions. We consider the infinite state space  $E = \{0, 1, 2, \dots\}$ . Denote by  $g$  and  $\text{Degree}(g)$ , respectively, the eigenfunction of  $\lambda_1$  and the degree of  $g$  when  $g$  is polynomial. Three examples of the perturbation of  $\lambda_1$  and  $\text{Degree}(g)$  are listed in Table 1.1.

**Table 1.1** Three examples of the perturbation of  $\lambda_1$  and  $\text{Degree}(g)$

$b_i (i \geq 0)$	$a_i (i \geq 1)$	$\lambda_1$	<b>Degree</b> ( $g$ )
$i + c (c > 0)$	$2i$	1	1
$i + 1$	$2i + 3$	2	2
$i + 1$	$2i + (4 + \sqrt{2})$	3	3

The first line is the well-known linear model, for which  $\lambda_1 = 1$ , independent of the constant  $c > 0$ , and  $g$  is linear. Next, keeping the same birth rate,  $b_i = i + 1$ , the perturbation of the death rate  $a_i$  from  $2i$  to  $2i + 3$  (respectively,  $2i + 4 + \sqrt{2}$ ) leads to the change of  $\lambda_1$  from one to two (respectively, three). More surprisingly, the eigenfunction  $g$  is changed from linear to quadratic (respectively, cubic). For the intermediate values of  $a_i$  between  $2i$ ,  $2i + 3$ , and  $2i + 4 + \sqrt{2}$ ,  $\lambda_1$  is unknown, since  $g$  is nonpolynomial. As seen from these examples, the first eigenvalue is very sensitive. Hence, in general, it is very hard to estimate  $\lambda_1$ .

Hopefully, we have presented enough examples to show the extreme difficulties of the topic. Very fortunately, at last, we are able to present a complete solution to this problem in the present context. Please be patient; the result will be given only later.

For a long period, we did not know how to proceed. So we visited several branches of mathematics. Finally, we found that the topic was well studied in Riemannian geometry.

## 1.2 New variational formula for the first eigenvalue

### A story of estimating $\lambda_1$ in geometry

Here is a short story about the study of  $\lambda_1$  in geometry.

Consider the Laplacian  $\Delta$  on a connected compact Riemannian manifold  $(M, g)$ , where  $g$  is the Riemannian metric. The spectrum of  $\Delta$  is discrete:  $\dots \leq -\lambda_2 \leq -\lambda_1 < -\lambda_0 = 0$  (may be repeated). Estimating these eigenvalues  $\lambda_k$  (especially  $\lambda_1$ ) is an important chapter in modern geometry. As far as



we know, five books, excluding books on general spectral theory, have been devoted to this topic: I. Chavel (1984), P.H. Bérard (1986), R. Schoen and S.T. Yau (1988), P. Li (1993), and C.Y. Ma (1993). About 2000 references are collected in the second quoted book. Thus, it is impossible for us to introduce an overview of what has been done in geometry. Instead, we would like to show the reader ten of the most beautiful lower bounds. For a manifold  $M$ , denote its dimension, diameter, and the lower bound of Ricci curvature by  $d$ ,  $D$ , and  $K$  ( $\text{Ricci}_M \geq Kg$ ), respectively. The simplest example is the unit sphere  $\mathbb{S}^d$  in  $\mathbb{R}^{d+1}$ , for which  $D = \pi$  and  $K = d - 1$ . We are interested in estimating  $\lambda_1$  in terms of these three geometric quantities. It is relatively easy to obtain an upper bound by applying a test function  $f \in C^1(M)$  to the classical variational formula

$$\lambda_1 = \inf \left\{ \int_M \|\nabla f\|^2 dx : f \in C^1(M), \int_M f dx = 0, \int_M f^2 dx = 1 \right\}, \quad (1.0)$$

where “ $dx$ ” is the Riemannian volume element. To obtain the lower bound, however, is much harder. In Table 1.2, we list ten of the strongest lower bounds that have been derived in the past, using various sophisticated methods.

**Table 1.2** Ten lower bounds of  $\lambda_1$

Author(s)	Lower bound
A. Lichnerowicz (1958)	$\frac{d}{d-1} K, \quad K \geq 0 \quad (1.1)$
P.H. Bérard, G. Besson, & S. Gallot (1985)	$d \left\{ \frac{\int_0^{\pi/2} \cos^{d-1} t dt}{\int_0^{D/2} \cos^{d-1} t dt} \right\}^{2/d}, \quad K = d - 1 > 0 \quad (1.2)$
P. Li & S.T. Yau (1980)	$\frac{\pi^2}{2D^2}, \quad K \geq 0 \quad (1.3)$
J.Q. Zhong & H.C. Yang (1984)	$\frac{\pi^2}{D^2}, \quad K \geq 0 \quad (1.4)$
D.G. Yang (1999)	$\frac{\pi^2}{D^2} + \frac{K}{4}, \quad K \geq 0 \quad (1.5)$
P. Li & S.T. Yau (1980)	$\frac{1}{D^2(d-1) \exp[1 + \sqrt{1 + 16\alpha^2}]}, \quad K \leq 0 \quad (1.6)$
K.R. Cai (1991)	$\frac{\pi^2}{D^2} + K, \quad K \leq 0 \quad (1.7)$
D. Zhao (1999)	$\frac{\pi^2}{D^2} + 0.52K, \quad K \leq 0. \quad (1.8)$
H.C. Yang (1990) & F. Jia (1991)	$\frac{\pi^2}{D^2} e^{-\alpha}, \quad \text{if } d \geq 5, \quad K \leq 0 \quad (1.9)$
H.C. Yang (1990) & F. Jia (1991)	$\frac{\pi^2}{2D^2} e^{-\alpha'}, \quad \text{if } 2 \leq d \leq 4, \quad K \leq 0 \quad (1.10)$