PARTIAL DIFFERENTIAL EQUATIONS

Theory and Technique

GEORGE F. CARRIER

CARL E. PEARSON

PARTIAL DIFFERENTIAL EQUATIONS

Theory and Technique

GEORGE F. CARRIER

Harvard University
Cambridge, Massachusetts

CARL E. PEARSON

University of Washington
Seattle, Washington

ACADEMIC PRESS New York San Francisco London 1976

A Subsidiary of Harcourt Brace Jovanovich, Publishers

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1

Library of Congress Cataloging in Publication Data

Carrier, George F
Partial differential equations.

Bibliography: p.
Includes index.
1. Differential equations, Partial—Numerical solutions. I. Pearson, Carl E., joint author. II. Title.
QA374.C36 515'.353 75-13107
ISBN 0-12-160450-0

AMS (MOS) 1970 Subject Classifications: 35-01, 35-02

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

This book reflects the authors' experience in teaching partial differential equations, over several years, and at several institutions. The viewpoint is that of the user of mathematics; the emphasis is on the development of perspective and on the acquisition of practical technique.

Illustrative examples chosen from a number of fields serve to motivate the discussion and to suggest directions for generalization. We have provided a large number of exercises (some with answers) in order to consolidate and extend the text material.

The reader is assumed to have some familiarity with ordinary differential equations of the kind provided by the references listed in the Introduction. Some background in the physical sciences is also assumed, although we have tried to choose examples that are common to a number of fields and which in any event are intuitively straightforward.

Although the attitudes and approaches in this book are solely the responsibility of the authors, we are indebted to a number of our colleagues for useful suggestions and ideas. A note of particular appreciation is due to Carolyn Smith, who patiently and meticulously prepared the successive versions of the manuscript, and to Graham Carey, who critically proofread most of the final text.

CONTENTS

INTRODUCTION

We collect here some formal definitions and notational conventions. Also, we analyze a preliminary example of a partial differential equation in order to point up some of the differences between ordinary and partial differential equations.

The systematic discussion of partial differential equations begins in Chapter 1. We start with the classical second-order equations of diffusion, wave motion, and potential theory and examine the features of each. We then use the ideas of characteristics and canonical forms to show that any second-order linear equation must be one of these three kinds. First-order linear and quasi-linear equations are considered next, and the first half of the book ends with a generalization of previous results to the case of a larger number of dependent or independent variables, and to sets of equations.

Included in the second half of the book are separate chapters on Green's functions, eigenvalue problems, and a more extensive survey of the theory of characteristics. Much of the emphasis, however, is on practical approximation techniques; attention is directed toward variational methods, perturbations (regular and singular), difference equations, and numerical methods.

I.1 DEFINITIONS AND EXAMPLES

A *partial differential equation* is one in which there appear partial derivatives of an unknown function with respect to two or more independent variables. A simple example of such an equation is

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - \sigma u = 0 \tag{I.1}$$

where σ is a constant. By a *solution* of this equation in a region R of the (x, y) plane we mean a function $u(x, y)$ for which u, $\partial u/\partial x$, and $\partial u/\partial y$ are defined at each point (x, y) in R and for which the equation reduces to an identity at each such point. Such a function u is said to *satisfy* the equation in R.

We denote partial derivatives by subscripts, so that $u_x = \partial u/\partial x$, $u_{xx} = \partial^2 u/\partial x^2$, $u_{xy} = \partial^2 u/\partial x\,\partial y$, etc. Other examples of partial differential equations are

$$x^2 u_{xx} + u_{xy} - \pi^2 u_{yy} + 3u_x - u = e^{x+y} \tag{I.2}$$

$$u_{xxy} - uu_x + \sin(xu^2) = 0 \tag{I.3}$$

$$u_{xyzz} + 2u_{zz} - u = \sin(x^2 + yz) \tag{I.4}$$

[In Eq. (I.4), u is a function of the three variables x, y, z.] Since the highest-order partial derivative that occurs in Eq. (I.1) is the first, Eq. (I.1) is said to be a *first-order* equation. Similarly, Eqs. (I.2), (I.3), and (I.4) are of the second, third, and fourth orders, respectively.

An important property that a partial differential equation may or may not possess is that of *linearity*. By definition, a linear partial differential equation for $u(x, y)$ has the form

$$\sum_{n=0}^{N} \sum_{m=0}^{M} a_{nm}(x, y) \frac{\partial^{n+m} u}{\partial x^n\, \partial y^m} = g(x, y) \tag{I.5}$$

where $a_{nm}(x, y)$ and $g(x, y)$ are given functions of x and y, and where N, M are fixed positive integers. (We define $\partial^0 u/\partial x^0\,\partial y^0$ to equal u.) If $g(x, y) \equiv 0$, we say that Eq. (I.5) is *homogeneous*. As with ordinary differential equations, the applicability of the principle of superposition is what makes linearity a useful property. Let $U(x, y)$ be one solution of Eq. (I.5), and let each of a set of functions $u^{(1)}(x, y)$, $u^{(2)}(x, y), \ldots,$ $u^{(p)}(x, y)$ be solutions of the homogeneous counterpart of Eq. (I.5); i.e.,

$$\sum_{n=0}^{N} \sum_{m=0}^{M} a_{nm}(x, y) \frac{\partial^{n+m} u^{(j)}}{\partial x^n\, \partial y^m} = 0, \qquad j = 1, 2, \ldots, p$$

Then if $a^{(1)}, a^{(2)}, \ldots, a^{(p)}$ are any p chosen constants, direct substitution into Eq. (I.5) shows that

$$u = U + a^{(1)}u^{(1)} + a^{(2)}u^{(2)} + \cdots + a^{(p)}u^{(p)}$$

is also a solution of Eq. (I.5).

Thus, Eqs. (I.1), (I.2), and (I.4) are linear, whereas Eq. (I.3) is nonlinear. Only rarely can one make much formal progress with nonlinear equations; fortunately, many equations of practical interest turn out to be linear (or almost linear).

Just as with an ordinary differential equation, many questions can be asked in connection with an equation such as (I.1). For example, (1) what function or functions, if any, satisfy Eq. (I.1) when $\sigma = 1$? (2) For what values of σ does a function $u(x, y)$ exist that satisfies Eq. (I.1)? (3) How many functions satisfy Eq. (I.1) in $y > 0$, $-\infty < x < \infty$ if we also require that $u(x, 0) = x^2$ for x in the interval $(0, 1)$?

In contrast to most of the partial differential equations we will encounter, Eq. (I.1) is rather easy to solve explicitly. In fact, for any value of σ we can define a new function $\phi(x, y)$ via

$$u = \phi \cdot \exp[\tfrac{1}{2}\sigma(x + y)]$$

(noting that the exponential factor is always nonzero); then $\phi(x, y)$ satisfies the equation

$$\phi_x + \phi_y = 0 \qquad (I.6)$$

With the change in variables $\xi = x + y$, $\eta = x - y$, and with

$$\psi(\xi, \eta) = \phi\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right)$$

[i.e., $\phi(x, y) = \psi(\xi, \eta)$ at corresponding points (x, y) and (ξ, η)], Eq. (I.6) becomes

$$2\psi_\xi = 0$$

so that ψ is a function of η alone, say $f(\eta)$. Since $\eta = x - y$, we can say equivalently that ϕ must be a function of $(x - y)$ alone. Thus u must have the form

$$u = f(x - y) \cdot \exp[\tfrac{1}{2}\sigma(x + y)] \qquad (I.7)$$

where f is an as-yet-undetermined function of the argument $(x - y)$. Conversely, the reader should show that if we choose any continuously differentiable function f and define a function u by Eq. (I.7), then u will satisfy Eq. (I.1).

The reader may now answer such questions as those posed above. In particular, the answer to question (3) can be found by use of Eq. (I.7). At $y = 0$, we have

$$u(x, 0) = f(x) \cdot \exp[\tfrac{1}{2}\sigma x]$$

and if this is to equal x^2 for x in $(0, 1)$, we must choose $f(x)$ such that

$f(x) = x^2 \exp[-\frac{1}{2}\sigma x]$ for x in $(0, 1)$. Replacing the argument x by $x - y$, it follows that

$$f(x - y) = (x - y)^2 \exp[-\tfrac{1}{2}\sigma(x - y)] \tag{I.8}$$

for $0 < x - y < 1$. Thus in that region of the (x, y) plane lying between the lines $y = x$ and $y = x - 1$, Eq. (I.7) yields

$$u = (x - y)^2 e^{\sigma y} \tag{I.9}$$

Outside the strip $0 < x - y < 1$, $f(x - y)$ can be any continuously differentiable function of $x - y$ that, at $x - y = 0$ and $x - y = 1$, joins continuously and with continuous first derivatives onto the function described by Eq. (I.8). The answer to question (3) is therefore that there are infinitely many solutions.

Notice that the general solution (I.7) involves an undetermined *function*, rather than simply an undetermined *constant*, as would be the case for a typical first-order ordinary differential equation. We can therefore anticipate that to determine completely the solution to an equation such as (I.1) we will have to specify u along some curve, rather than merely at a single point. Moreover, even such a specification of the solution along a curve may determine the solution only within a region determined by that curve, as in the example just discussed.

The above discussion can be generalized in several ways. Instead of only two independent variables x and y, we may have a number of such variables, and instead of only one dependent variable u, there may be a number of such functions to be determined. We will let a single example suffice. If u and v are each functions of (x, y, z), then the equations

$$uu_x + uu_y + u_z v_z = 1$$

$$x^2 u_{xx} + u_y + v_{zz} = \sin(x + u)$$

would form a coupled pair of nonlinear equations for u and v.

As a different kind of generalization, we can weaken the term "solution" as defined in the first paragraph. It may be physically reasonable to permit a particular derivative, for example, to be discontinuous at a certain point or along a certain curve in the (x, y) plane, and perhaps even greater liberties with the idea of a "solution" can be taken when they are consistent with the context in which a problem arises. We shall encounter such situations later in this book, but in the early sections the given definition is to apply unless an alternative is explicitly stated.

The subject of partial differential equations is a broad one, and it seems useful to begin by acquiring experience with certain frequently encountered special equations. This we will do in the next few chapters.

In so doing we shall focus attention primarily on the techniques by which equations are generated (as a result of model-building) and by which solutions are found, and on the features that characterize these equations and their solutions.

Throughout, it will be assumed that the reader is familiar with, or can easily refer to, such properties of ordinary differential equations as are discussed in standard texts.† The abbreviations ODE and PDE will sometimes be used for "ordinary differential equation" and "partial differential equation," respectively. When particular attention is to be directed to a continuity property, the notation $C^{(n)}$ may be used to indicate continuity of nth derivatives. When no specification to the contrary is made, it is to be understood that boundary curves or surfaces are smooth, in the sense of having continuously turning tangents, and that functional data specified on such boundaries are continuous.

The problems are considered to be an integral part of the text. The reader who evades them will miss 72% of the value of the book.

† A representative selection follows: Kreyszig (1967, Chaps. 1–4); Boyce and DiPrima (1969); Coddington (1961); Birkhoff and Rota (1962); Carrier and Pearson (1968); Ince (1956); Kamke (1948; this text contains a dictionary of solutions).

1

THE DIFFUSION EQUATION

1.1 DERIVATION

One of the more common partial differential equations of practical interest is that governing diffusion in a homogeneous medium; it arises in many physical, biological, social, and other phenomena. A simple example of such an equation is

$$\phi_t = a^2\phi_{xx} \tag{1.1}$$

Here x is position, t time, a a positive constant,† and we seek a function $\phi(x, t)$ satisfying this equation for a certain range of x and t values. In addition, ϕ is usually required to satisfy certain auxiliary conditions.

Much of our attention in this chapter will be directed toward Eq. (1.1)—the one-dimensional diffusion equation with constant coefficients. However, before considering properties of the equation itself, it seems worthwhile to derive it (with reasonable care) in at least one context in which it arises. We choose the problem of heat flow along a thin rod with insulated sides, since the associated physics is rudimentary.

Let the rod be oriented along the x-axis; denote its cross-sectional area by A, its density by ρ, its specific heat by c, and its thermal conductivity by k. We take the temperature ϕ (measured relative to some chosen reference level) as being a function of x and t only, i.e., ϕ has the same value at each point in any chosen cross section. To start with, we restrict ourselves

† We write the coefficient as a^2 for future convenience.

to the case in which each of A, ρ, c, and k is a constant (and so is independent of x, t, or ϕ).

Let us single out for consideration a portion of the rod lying between any two points α and β, with $\beta > \alpha$. From the definition of specific heat the rate at which thermal energy is accumulating within this portion of the rod is

$$R_1 = \int_\alpha^\beta \phi_t(x, t) c\rho A \, dx \tag{1.2}$$

However, heat is transported by diffusion in the direction of, and at a rate proportional to, the negative of the temperature gradient. Thus, the net rate R_2 at which heat enters the segment $\alpha < x < \beta$ is

$$R_2 = kA\phi_x(\beta, t) - kA\phi_x(\alpha, t) \tag{1.3}$$

Clearly, $R_1 = R_2$. Hence

$$\int_\alpha^\beta \phi_t(x, t) c\rho A \, dx = kA\phi_x(\beta, t) - kA\phi_x(\alpha, t)$$

$$= kA \int_\alpha^\beta \phi_{xx}(x, t) \, dx$$

It follows that

$$\int_\alpha^\beta [\phi_t c\rho A - kA\phi_{xx}] \, dx = 0 \tag{1.4}$$

But α and β were arbitrarily chosen positions along the rod. Equation (1.4) can hold for *any* choice of α and β only if the integrand is identically zero.† This implies then that

$$\phi_t c\rho A - kA\phi_{xx} = 0$$

or

$$\phi_t = a^2 \phi_{xx} \tag{1.5}$$

where $a^2 = k/(c\rho)$ is termed the *thermal diffusivity*.

† The basic theorem to which we appeal here is as follows. Let $\psi(x)$ be a continuous function of x satisfying the condition that $\int_\alpha^\beta \psi(x)\,dx = 0$ for all choices of α and β in some interval. Then $\psi(x) \equiv 0$ in that interval. For otherwise, $\psi(x)$ would be nonzero at some point x_0 in the interval, and in consequence of continuity, $\psi(x)$ would be nonzero and would have the same sign as $\psi(x_0)$ in some small subinterval around x_0; a choice of α and β within this subinterval would then lead to a nonzero value of the integral, which provides a contradiction.

To use this theorem, we require that the integrand of Eq. (1.4) be continuous, and for temperature this is a reasonable physical expectation.