Theoretical Electromagnetism

R. H. ATKIN

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by

R. H. ATKIN
M.A. (Cantab.), M.Sc. (London)
Principal of Unischol College of Science, London

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Preface

This book is about electrostatics, magnetostatics, and the electromagnetic field. I hope that it will be of use to students of theoretical physics, whether they be physicists or mathematicians, and I have concentrated on building up the mathematical structure in as concise a manner as possible.

The appeal to experimental facts has been made in the beginning and consists only of a recital of the conventional essentials. In this respect I must ask the reader to tolerate the break from a certain tradition, viz. the usual accounts of the "fur-ebonite-lodestone" experiments of the Ancient Greeks and their later medieval pupils.

It is important for the reader to have some knowledge of vector field theory, certain theorems in complex function theory, and the concept of a tensor and its matrix representation. The necessary theorems are collected in review in the appropriate chapters, but are not otherwise discussed.

I have again offered solutions to a large number of typical problems which have been carefully selected from examination papers set in the universities of Cambridge, Oxford and London. I would like to express my gratitude to these authorities for permission to publish the examination questions: the solutions are my own responsibility.

I would also like to express my appreciation of the many valuable suggestions made by Dr. C. W. Kilminster, who was kind enough to read the book in manuscript.

Finally I would like to thank the publishers for their constant encouragement during the writing of this book.

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CHAPTER I

The Foundations of Electrostatics

1.1 Some preliminary ideas

The object of our study in the following chapters is the "world of electricity and magnetism", as discovered by experiment and as made systematic and aesthetic by the language and techniques of mathematics.

In the first instance we assume the existence of a world defined by the *electric charge* and the *electric field.* This is comparable with the material world which is defined (to some extent at least) by the *gravitational matter* and the *gravitational field.* Thus we agree to say that it is experimentally possible to obtain material bodies possessing the additional property of being *electrically charged*; the electric charge is a *scalar* quantity, occurring as either *positive charge* or as *negative charge*; these charged bodies exert forces (line vectors) of attraction/repulsion upon each other according to the formula "like charges repel, unlike charges attract"; these electric charges therefore generate an *electric field vector* at all points of space (in vacuo). Furthermore, under the title of Electrostatics, we are only concerned with charges which are in equilibrium under the action of their mutual field.

Notice that we are saying that material bodies *possess* charge; we are not thinking of "pieces of charge" in the same way as we think of pieces of metal. Ideas which are concerned with chasing the electric charge until it exists (experimentally) independently of matter automatically lead us to Atomic Physics. If the reader finds it helpful then we can remark that the basic unit of negative electricity appears to be what is called an *electron* and this is associated in the atom with an equal (numerically) positive charge in the shape of a *proton.* Thus Atomic Physics has taken (e.g.) the hydrogen atom, which in its normal state gives the appearance of being electrically neutral, and separated it into two energy components (the electron and the proton) each of which exhibits the symptoms of electric charge. From our point of view, which is a *macroscopic* view, the only idea which we need to borrow from atomic physics is that atoms are normally neutral but that they can be made to exhibit a positive and a negative electrical charge.

We shall use the letter e to denote charge and this is assumed to be positive unless explicitly stated to be otherwise. The electric field vector will be denoted by $\boldsymbol{E}$, so that if a *small body* carrying a *small charge* e is introduced into a region where $\boldsymbol{E} \neq \boldsymbol{O}$ it will experience a force $\boldsymbol{F}$ given by

$$\boldsymbol{F} = e\boldsymbol{E} \qquad \ldots.(1)$$

Thus $\boldsymbol{E}$ is taken as the force experienced by a unit positive charge at the point in question. Of course $\boldsymbol{E}$ is a *vector point function* and should perhaps be written as $\boldsymbol{E}(\boldsymbol{r})$. This is why we speak of a *small* charged body in the field—to ensure that $\boldsymbol{E}(\boldsymbol{r})$ has a constant value at all points of the body. This is patently not possible for macroscopic bodies so that equation (1) is already a convenient mathematical device. To emphasise the conditions under which (1) is true we shall commonly refer to a *point-charge* e instead of using the words "small charged body". We shall also have occasion to regard e itself as a *scalar point function* $e(\boldsymbol{r})$, but in this case we shall speak of *charge density* $\rho(\boldsymbol{r})$ per unit volume, or $\sigma(\boldsymbol{r})$ per unit area. When we use the notation $\sigma(\boldsymbol{r})$ for the charge density per unit area we shall obviously be referring to a charged surface and so we speak of the *surface charge density* $\sigma(\boldsymbol{r})$.

It is often said that there is an analytical difficulty in the use of scalar point functions $\rho(\boldsymbol{r})$ and $\sigma(\boldsymbol{r})$ since they cannot be continuous—appeal being made to the discrete nature of charges in the form of electrons and protons. In the author's view this is greatly over-estimated, the following points being regularly overlooked:

(i) even assuming $\rho(\boldsymbol{r})$ is not continuous it does not involve difficulties in integration (the common problem) provided it is defined and bounded in the region of integration.

(ii) if in fact $\rho(\boldsymbol{r})$ is really the charge function due to the presence of electrons or protons then, since we have borrowed from atomic physics so far, why not borrow some more and say that the charge density can be represented by the value of $|\psi(\boldsymbol{r})|^2$ where $\psi(\boldsymbol{r})$ is the scalar wave function of Quantum Mechanics; of course $\psi(\boldsymbol{r})$ is continuous.*

Conductors and insulators

For the purposes of Electrostatics we assume that substances can be divided into the above two classes.

If a charge is inside (or on) a conductor then the application of a field $\boldsymbol{E}$ *causes the charge to move freely; there is no restriction on the movement of the charge.*

The above sentence defines the class of conductors.

A substance is an insulator if it is not a conductor; the class of insulators is the complement of the class of conductors.

In practice it is better to think of good conductors and bad conductors rather than conductors and insulators since the difference in the laboratory is really one of degree. Our definitions constitute an attempt to divide all substances into the two idealized types, viz. *perfect conductors* or *perfect non-conductors*. We must emphasize that this is acceptable in the study of Electrostatics but is not appropriate for Electrodynamics (Ch. III *et seq.*).

* See the author's book "Mathematics and Wave Mechanics" for a full discussion of $\psi(\boldsymbol{r})$.

The material of an insulator is called a *dielectric* material, and when an electric field is applied to such a substance we agree to say that the positive and negative charges which compose the neutral atoms separate very slightly under the action of the field. A state of internal stress therefore appears; we call the process *polarisation*.

In the case of a conductor the picture is very different. The application of an electric field causes the complete separation of positive and negative atomic charges so that all the positive charge moves to one end of the body and all the negative to the other end. This process is called *induction* and commonly appears if a charged body is brought near to an insulated conductor.*

Finally we need to acknowledge the following *experimental facts:*
In Electrostatics (charge is in equilibrium) we find

(i) if $\rho(\boldsymbol{r}) \equiv 0$ inside a hollow conductor then $\boldsymbol{E} \equiv \boldsymbol{O}$ at all points inside,

(ii) there is no free electric charge inside the substance of a conductor. ....(2)

From the definition of a conductor, and from (2), it follows that *all the charge resides on the surface of a conductor*, and the *field vector* ***E*** *is normal to the surface of a charged conductor*—if $\boldsymbol{E}$ had a tangential component the charge would move.

Returning to the idea of the general electric field defined by the line vector $\boldsymbol{E}(\boldsymbol{r})$ we must regard it as fundamental that **$\boldsymbol{E}(\boldsymbol{r})$ defines a conservative field**.† Thus we can say immediately that a scalar point function $V(\boldsymbol{r})$ exists at all points where $\boldsymbol{E}$ exists and is given by

$$\left.\begin{aligned} \boldsymbol{E} &= -\text{grad } V \\ \text{or} \quad \boldsymbol{E} &= -\nabla V \end{aligned}\right\} \qquad \ldots.(3)$$

Since $V(\boldsymbol{r})$ must be differentiable at all points where $\boldsymbol{E}$ is defined then V possesses the additional property

$$V(\boldsymbol{r}) \text{ is continuous} \qquad \ldots.(4)$$

1.2 The mathematical framework

It will be necessary to assume an acquaintance on the part of the reader with the basic formulae of Vector Field Theory. These ideas are developed in the author's "Mathematics and Wave Mechanics" (abbreviated for future reference to "M and WM") while the role of

* The reader who is interested in the practical experimental details of electrostatic charges should study (e.g.) A. S. Ramsey, "Electricity and Magnetism" which gives a full account of the behaviour of the electroscope, etc.

† See the author's book Classical Dynamics (Ch. III) for a full discussion of conservative field.

vector algebra and tensors will be found in "Classical Dynamics" (abbreviated to "CD").

For reference we shall find it valuable to compile the following list of formulae and theorems.

If $V(\boldsymbol{r})$ is a differentiable scalar point function its increment consequent upon $\boldsymbol{r}$ changing to $\boldsymbol{r} + d\boldsymbol{r}$ is given by

$$dV = (\text{grad } V) \,.\, d\boldsymbol{r}$$

the dot signifying scalar product. Using the nabla operator ∇ we can write

$$dV = (\nabla V) \,.\, d\boldsymbol{r} \qquad \ldots .(1)$$

where

$$\nabla V = \frac{\partial V}{\partial x_1} \boldsymbol{e}_1 + \frac{\partial V}{\partial x_2} \boldsymbol{e}_2 + \frac{\partial V}{\partial x_3} \boldsymbol{e}_3 \qquad \ldots .(2)$$

if $\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$ and $(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)$ being an orthonormal triad of reference. When we use (x, y, z) in place of $(x_1x_2x_3)$ it is usual to write $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ in place of $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. Thus with respect to $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ the operator ∇ becomes

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \qquad \ldots .(3)$$

If $\boldsymbol{F}(\boldsymbol{r})$ is a vector point function with differentiable components we define the *divergence* and *curl* of $\boldsymbol{F}$ by

$$\text{div } \boldsymbol{F} = \nabla \,.\, \boldsymbol{F} \qquad \ldots .(4)$$

$$\text{curl } \boldsymbol{F} = \nabla \times \boldsymbol{F} \qquad \ldots .(5)$$

With respect to the usual $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ these become

$$\text{div } \boldsymbol{F} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \qquad \ldots .(6)$$

and

$$\text{curl } \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix} \qquad \ldots .(7)$$

or

$$\text{curl } \boldsymbol{F} = \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\boldsymbol{k} \qquad \ldots .(8)$$

From (2) and (6) it follows that

$$\text{div grad } V = \nabla \,.\, (\nabla V) = \nabla^2 V$$

and

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \qquad \ldots .(9)$$

We shall require the theorems of Gauss and of Stokes, viz.,

Gauss' Theorem

Given a closed surface S enclosing a volume Ω of space, and a vector point function $\boldsymbol{F}(\boldsymbol{r})$*, then*

$$\int_\Omega \operatorname{div} \boldsymbol{F}\, d\Omega = \int_S \boldsymbol{F} \,.\, d\boldsymbol{S} \qquad \ldots.(10)$$

where $d\boldsymbol{S}$ means $\boldsymbol{n}\, dS$, $\boldsymbol{n}$ being the unit vector in the direction of the outward normal to S at dS.

Stokes' Theorem

Given a closed curve C spanned by a surface S, and a vector point function defined at all points of S, and possessing continuous first order derivatives, then

$$\int_S (\operatorname{curl} \boldsymbol{F}) \,.\, d\boldsymbol{S} = \oint_C \boldsymbol{F} \,.\, d\boldsymbol{s} \qquad \ldots.(11)$$

where $d\boldsymbol{s}$ means $\boldsymbol{t}\, ds$, $\boldsymbol{t}$ being the unit vector in the direction of the tangent to C and ds the element of arc of C.

Curvilinear co-ordinates. When the co-ordinate system remains orthogonal but the metric* takes the more general form of

$$ds^2 = h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2 \qquad \ldots.(12)$$

the components of div, grad and curl become altered according to the following scheme.

$$\operatorname{grad} f = \nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\boldsymbol{e}_3 \qquad \ldots.(13)$$

$$\operatorname{div} \boldsymbol{F} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(h_2h_3F_1) + \frac{\partial}{\partial u_2}(h_3h_1F_2) + \frac{\partial}{\partial u_3}(h_1h_2F_3)\right\}$$

where $\boldsymbol{F} = F_1\boldsymbol{e}_1 + F_2\boldsymbol{e}_2 + F_3\boldsymbol{e}_3$ ....(14)

Also

$$\operatorname{curl} \boldsymbol{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} \qquad \ldots.(15)$$

* See M and WM, Ch. VII.

Combining (13) and (14) we get

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \cdot \left\{ \frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right\} \quad \ldots.(16)$$

The following identities can be easily verified from the above definitions

$$\text{div curl } \boldsymbol{F} = O \qquad \text{for all } \boldsymbol{F} \quad \ldots.(17)$$

$$\text{curl grad } V = O \qquad \text{for all } V \quad \ldots.(18)$$

$$\text{curl curl } \boldsymbol{F} = \text{grad div } \boldsymbol{F} - \nabla^2 \boldsymbol{F} \quad \ldots.(19)$$

where $\nabla^2 \boldsymbol{F} = \left(\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} + \frac{\partial^2 X}{\partial z^2}\right)\boldsymbol{i}$ + similar expressions

i.e. $$\nabla^2 \boldsymbol{F} = (\nabla^2 X)\boldsymbol{i} + (\nabla^2 y)\boldsymbol{j} + (\nabla^2 Z)\boldsymbol{k} \quad \ldots.(20)$$

In particular, using *cylindrical polar coordinates* (r, θ, z) with metric

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

(16) gives $$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \quad \ldots.(21)$$

and in two dimensions (r, θ) this reduces to

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \quad \ldots.(22)$$

When using spherical polar coordinates (r, θ, ϕ) with metric

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

(16) gives

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \quad \ldots.(23)$$

and when f is independent of ϕ (i.e. the axis $\theta = 0$ is an axis of symmetry) this becomes

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) \quad \ldots.(24)$$

1.3 The laws of Gauss and of Coulomb

Coulomb's Law is really the inverse-square law: *two point charges e_1 and e_2 a distance r apart repel each other with a force $\frac{e_1 e_2}{r^2}$, in vacuo.*

Thus, the law states that the electric field vector due to a charge $+e$ at the origin is (v. Fig. 1)

$$E = \frac{e}{r_3} \mathbf{r} \qquad \ldots.(1)$$

Such a law involves the concept of "action at a distance" and if it is thought desirable to avoid this then we must have recourse to Gauss' Law, viz.

If S is a closed surface enclosing a total charge $+e$ then the total outward flux of **E** *over S equals $4\pi e$,* that is to say

$$\int_S \mathbf{E} \,.\, d\mathbf{S} = 4\pi e \qquad \ldots.(2)$$

Now either (1) or (2) may be taken as the starting point of electrostatics, Gauss' Law being preferable since it does not involve the idea of action at a distance; we may imagine the effect of the charge e spreading outwards through space by slowly increasing S in size and shape.

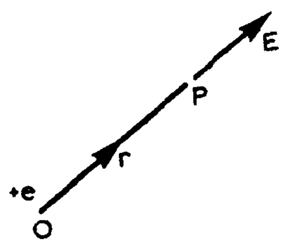

Fig. 1

Example 1. *To deduce Coulomb's Law from Gauss' Law.* Using Fig. 1, imagine a sphere S, centre 0 and radians r. Then by symmetry E will be a function of r only and therefore a constant at all points P of S. Applying (2) we get

$$\int_S \mathbf{E} \,.\, d\mathbf{S} = E \,.\, 4\pi r^2 = 4\pi e$$

so that $E = \frac{e}{r^2}$ as required.

Example 2. *To deduce Gauss' Law from Coulomb's Law.* Let S be any surface completely enclosing a point charge $+e$ at the origin 0. Then the total outward flux over S will be

$$\int_S \mathbf{E} \,.\, d\mathbf{S} = e\int_S \frac{1}{r^3} \mathbf{r} \,.\, d\mathbf{S} = e\int_S -\nabla\left(\frac{1}{r}\right) . \, d\mathbf{S}$$

$$= e\int_S d\omega$$

where $d\omega$ is the solid angle subtended at 0 by the element of area $d\mathbf{S}$.* Hence, since the total solid angle at a point is 4π, the result follows.

* See M and WM, p. 170.

The electric potential function $V(r)$ due to a point-charge e at the origin now follows from (1).

We have
$$E = \frac{e}{r^3}\mathbf{r} = -\nabla V$$

and since
$$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\mathbf{r} \quad \text{we deduce}$$

$$V(r) = \frac{e}{r} \qquad \ldots.(3)$$

If the surface S becomes one which encloses a charge distribution $\rho(\mathbf{r})$ throughout a volume Ω then Gauss' Law becomes

$$\int_S \mathbf{E} \,.\, d\mathbf{S} = 4\pi \int \rho(\mathbf{r})\, d\Omega \qquad \ldots.(4)$$

and the potential at a point P *outside* Ω becomes

$$V(P) = \int_\Omega \frac{\rho(\mathbf{r})}{r}\, d\Omega \qquad \ldots.(5)$$

where r is measured from P to $d\Omega$.

If P is inside Ω the integrand becomes infinite although we can show that the integral converges. Thus we regard Ω as composed of Ω_0 and Ω_1 where Ω_1 contains P as an interior point and is enclosed by a simple surface Σ. Also $\rho(\mathbf{r})$ must be regarded as defined and bounded at all points of Ω; nor is there any loss of generality in taking ρ to be positive at all points of Ω.

Now let Σ be entirely contained between the concentric spheres, centre P, of radii a and b $(b > a)$; let $\rho \geq \rho_1$, in $r \leq a$ and $\rho \leq \rho_2$ in $r \leq b$. Then we need to show that

$$\lim_{\Omega_1 \to 0} \int_{\Omega_1} \frac{\rho}{r}\, d\Omega = 0 \qquad \ldots.(6)$$

Writing
$$F = \int_{\Omega_1} \frac{\rho}{r}\, d\Omega$$

we have
$$4\pi\rho_1 \int_0^a \frac{1}{r} \,.\, r^2\, dr \leq F \leq 4\pi\rho_2 \int_0^b \frac{1}{r} \,.\, r^2\, dr$$

that is,
$$2\pi\rho_1 \,.\, a^2 \leq F \leq 2\pi\rho_2 \,.\, b^2$$

As $a, b \to 0$ equation (6) follows.

Units. Coulomb's Law (1) enables us to define the *electrostatic unit of charge* (e.s.u.) as the common value of e_1 and e_2 when $r = 1$ cm. and the force is 1 dyne. Similarly the e.s.u. of potential difference is such that 1 erg of energy is required to move 1 e.s.u. of charge through that difference.

These units are not however the practical units which are as follows:

$$1 \text{ coulomb} = 3 \times 10^9 \text{ e.s.u. of charge}$$

$$1 \text{ volt} = \frac{1}{300} \text{ e.s.u. of p.d.}$$

1.4 Field equations

If we apply Gauss' Theorem of 1.2(10) to Gauss' Law of 1.3(4) we get, for all surfaces S,

$$\int_S \boldsymbol{E} \cdot d\boldsymbol{S} = \int_\Omega \operatorname{div} \boldsymbol{E}\, d\Omega = 4\pi \int \rho(\boldsymbol{r})\, d\Omega$$

i.e. for all Ω,

$$\int_\Omega (\operatorname{div} \boldsymbol{E} - 4\pi\rho)\, d\Omega = 0$$

Hence at all points of space

$$\operatorname{div} \boldsymbol{E} = 4\pi\rho \qquad \ldots.(1)$$

This, together with

$$\boldsymbol{E} = -\nabla V \qquad \ldots.(2)$$

constitute the electrostatic field equations, being true at all points of the field—with the proviso that, so far, we have not taken any account of the effect of the medium.

Combining (1) and (2) we obtain *Poisson's equation*

$$\nabla^2 V = -4\pi\rho \qquad \ldots.(3)$$

and at all points at which $\rho = 0$ this becomes *Laplace's equation*

$$\nabla^2 V = 0 \qquad \ldots.(4)$$

These equations (3) and (4), particularly the latter, contain the solution of a large range of electrostatic problems in one, two and three dimensions. Many of these problems consist of charged conductors in vacuo (or in a medium of simple isotropic electrical properties) and the information at our disposal amounts to knowledge of the boundary properties, e.g. the values of V and of $\frac{\partial V}{\partial n}$ over a specified surface. The method of solution in such cases consists of finding that solution of (4) which fits the appropriate boundary conditions. We shall investigate this general method of attack in the next chapter.

1.5 Lines of force and equipotential surfaces

A curve in space, Γ, is a *line of force* if its tangent defines the direction of $\boldsymbol{E}$ at every point of itself. Thus the equations of Γ are given by solving

$$\frac{dx}{E_1} = \frac{dy}{E_2} = \frac{dz}{E_3} \qquad \ldots.(1)$$

where $\boldsymbol{E} = E_1\boldsymbol{i} + E_2\boldsymbol{j} + E_3\boldsymbol{k}$.

If the lines of force are drawn through every peripheral point of a small area ΔS they can be regarded as all parallel and perpendicular to ΔS. We thus have what Faraday called a *tube of force* (Fig. 2). If S_1 and S_2 are areas of two cross sections of the same tube of force and

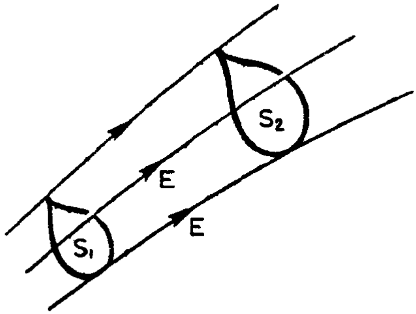

Fig. 2

E_1 and E_2 the electric intensities at these sections, then applying Gauss' Law to this section we get

$$\int_S \boldsymbol{E} \cdot d\boldsymbol{S} = 0$$

which is $$-E_1S_1 + E_2S_2 = 0$$

or, the "strength" of the tube ES is constant.

A line of force naturally passes from positive charge to negative charge (v. Fig. 1) since $\boldsymbol{E}$ leaves the one and arrives at the other. Thus a positive point charge free to move in space must travel along the lines of force. Of course there is likely to be a distortion of these lines due to the presence of the point charge itself.

An *equipotential surface* is one whose points are all at the same potential. Thus if $V(x, y, z)$ denotes the potential function in some region of space the equipotential surfaces will have equations

$$V(x, y, z) = \lambda \qquad \ldots.(2)$$

where λ is a parameter. If $P(x, y, z)$ lies on (2) then $V(P) = \lambda$. As λ changes (2) generates a whole family of equipotential surfaces—of course the separate potentials are different. Equation 1.3(3) shows that the equipotentials in the space occupied by a point charge at the origin are given by

$$r = \lambda \qquad \ldots.(3)$$

i.e. a set of concentric spheres.

Now (2) shows that for displacements $\boldsymbol{dr}$ in the surface itself (λ remains constant) we get

$$dV = (\nabla V) \,.\, \boldsymbol{dr} = 0$$

which is
$$\boldsymbol{E} \,.\, \boldsymbol{dr} = 0 \qquad \ldots .(4)$$

Equation (4) shows that the lines of force are everywhere orthogonal to the equipotential surfaces. Hence *the lines of force form a set of curves* $\{\Gamma\}$ *which are orthogonal to the equipotential family* $\{\Lambda\}$.

It follows from this that *a conductor must always be an equipotential surface in an electrostatic field.*

The following result is now important.

The condition for a family of surfaces $\{\mathscr{F}\}$ ***to be an equipotential family*** $\{\Lambda\}$.

Let $\{\mathscr{F}\}$ be defined by

$$f(x, y, z) = \lambda \qquad \ldots .(5)$$

where λ is a parameter. We require to find the condition which must be satisfied before we can say that $\{\mathscr{F}\}$ can be an equipotential family in some hypothetical electrostatic system. We must suppose therefore that the hypothetical potential function $V(x, y, z)$ will be a function of λ, that is, we can write $V = g(\lambda)$ where $g(\lambda)$ is a single-valued, continuous and differentiable function of λ. Since we require that $\nabla^2 V = 0$, we have

$$\frac{\partial V}{\partial x} = g'(\lambda)\frac{\partial \lambda}{\partial x}$$

and
$$\frac{\partial^2 V}{\partial x^2} = g''(\lambda)\left(\frac{\partial \lambda}{\partial x}\right)^2 + g'(\lambda)\frac{\partial^2 \lambda}{\partial x^2}$$

and so we require
$$g'(\lambda)\{\nabla^2 \lambda\} + g''(\lambda)\{\nabla \lambda\}^2 = 0$$

Using (5) this gives
$$\frac{\nabla^2 f}{(\nabla f)^2} = -\frac{g''(\lambda)}{g'(\lambda)}$$

i.e.
$$\frac{\nabla^2 f}{(\operatorname{grad} f)^2} = -\frac{d}{d\lambda}\{\log [g'(\lambda)]\} \qquad \ldots .(6)$$

which is the condition. The essential point in (6) is that $\dfrac{\nabla^2 f}{(\operatorname{grad} f)^2}$ *shall be a function of* λ *(only).*

Example 1. *Can the set of concentric spheres $r = \lambda$ be an equipotential family?*

Writing $f \equiv r = \lambda$ we have $\dfrac{\partial f}{\partial x} = \dfrac{x}{r}$

so that
$$\nabla f = \frac{x}{r} i + \frac{y}{r} j + \frac{z}{r} k$$

and
$$|\nabla f|^2 = \frac{x^2 + y^2 + z^2}{r^2} = 1$$

and also
$$\nabla^2 f = \frac{1}{r} - \frac{x^2}{r^3} + \frac{1}{r} - \frac{y^2}{r^3} + \frac{1}{r} - \frac{z^2}{r^3} = \frac{2}{r}$$

Hence
$$\frac{\nabla^2 f}{|\nabla f|^2} = \frac{2}{r} = \frac{2}{\lambda} = \text{a function of } \lambda$$

Also the potential function in this case will be $V = g(\lambda)$

where
$$-\frac{d}{d\lambda}\{\log g'(\lambda)\} = \frac{2}{\lambda}$$

that is
$$\log g'(\lambda) = -2 \log \lambda + \text{constant}$$

which gives
$$g'(\lambda) = a\lambda^{-2} \qquad (a = \text{constant})$$

and so
$$V = g(\lambda) = \frac{A}{r} + B$$

where A and B are constants. This of course is a result we expected. The next result will be found very useful.

Example 2. *Can the family of confocal ellipsoids*

$$\frac{x}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1$$

be an equipotential family?

To avoid the differentiation becoming too involved we shall adopt the notation

$$P = \frac{1}{a^2 + \lambda} + \frac{1}{b^2 + \lambda} + \frac{1}{c^2 + \lambda}$$

and
$$Q_n = \frac{x^2}{(a^2 + \lambda)^n} + \frac{y^2}{(b^2 + \lambda)^n} + \frac{z^2}{(c^2 + \lambda)^n}$$