Bifurcation Theory, Mechanics and Physics

edited by

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FOREWORD

This volume presents the proceedings of a colloquium inspired by the former President of the French Mathematical Society, Michel Herve. The aim was to promote the development of mathematics through applications.

Since the ancient supports the new, it seemed appropriate to center the theoretical conferences on new subjects.

Since the world is movement and creation, the theoretical conferences were planned on mechanics (movement) and bifurcation theory (creation). Five aspects of mechanics were to be presented, but, unfortunately, it has not been possible to include the statistical mechanics aspect. So that only four aspects are presented:

- Classical mechanics (Hamiltonian, Lagrangian, Poisson) (W.M. Tulczyjew, J.E. White, C.M. Marle).
- Quantum mechanics (in particular the passage from the classical to the quantum approach and the problem of finding the explicit solution of Schrödinger's equation)(M. Cahen and S. Gutt, J. Leray).
- Fluid mechanics (meaning problems involving partial differential equations. One of the speakers we hoped would attend the conference was in Japan at the time, however his lecture is presented in these proceedings.) (J.F. Pommaret, H.W. Shi)
- Mathematical "information" theory (S. Guiasu)

Traditional physical arguments are characterized by their great homogeneity, and mathematically expressed by the compactness property. In such cases, there is a kind of duality between locality and globality, which allows the use of the infinitesimal in global considerations.

In the papers, infinitesimal methods appear through the use of infinitesimal operators (in particular differential forms and Lie groups), and through the use of Taylor's series expansion (jet bundles: the use of this language is the most convenient in the search for solutions of partial differential equations on Riemannian spaces). Global considerations appear through the use of global energy functions, and extremal or variational principles (see the

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paper by L. Nirenberg). None of these principles is really well understood.

Three aspects of bifurcation theory are embraced:

- Bifurcation in ordinary differential equations (the lecturer centered his talk on his work on some special Hopf bifurcations. The results are included in the reviewed paper written by a colleague who could not attend the colloquium (W. Broer).
- Numerical methods in bifurcation theory tied to the Lyapunov-Schmidt procedure (J. Rappez).
- Bifurcation in partial differential equations involving the the Lyapunov-Schmidt procedure and singular theory (M. Golubitsky).

At this point, a feeling arises that group considerations play an increasing role in the study of bifurcation phenomena. These can be understood as the results of bifurcations of group (and pseudo-group) actions.

Monge's method, expanded by W. Shi, is used by W. H. Shi to treat a non-trivial example. The physical significance of this example is criticized by J. Leray; however the method remains strong despite this criticism. Besides I would like to point out that we are not always sure which mathematical formulation of a physical problem is the best.

This volume presents a large number of open problems concerning topological methods which are useful to show the influence of the topology of the space of solutions induced by the functional equations to be solved and the nature of the boundary conditions. The arising or vanishing of topological obstructions are obviously bound to shock and bifurcation phenomena. In any case, the Monge-Shi method has to be handled with care in order to take into account Borel's phenomenon (a well-chosen variational coefficient induces the non-analyticity of the unique solution of the partial differential equation with analytical data) or the turbulence phenomena. The problem already posed by small denominators in classical mechanics suggest that the role of number theory will increase in the study of refinements in bifurcation theory.

The last paper presented on bifurcation theory (M. Golubitsky) mainly concerns the Bénard problem. This paper is followed by an illustrative article (S. Fauve and A. Lichaber) on experiments showing turbulence phenomena and chaos. At that moment, the homogeneity of the physical state is somehow perturbed but through renormalization, a discrete modelization applies.

Numerical analysis can only use discrete models. Non-standard analysis can be seen as a convergence technique from the discrete to the non-discrete. The next paper (C. Lober and C. Reder) uses this recognition to solve some classical partial differential equations. Automata defined on finite sets (time excepted) can but admit periodic or quasi-periodic regimes. Extensions of such models through non-standard analysis might preserve some periodicities and be convenient models for some natural phenomena, including chemical systems.

The next paper (P. Dousson) is devoted to the mathematical study through quasi-autonomous "ordinary" differential or chemical systems satisfying Wei's axiomatic equations.

The volume concludes with papers (F.A. Grünbaum, M. Kleman, Y. Bouligand) on new mathematical applications to subjects which have recently been developed: tomography on the one hand, liquid crystals on the other hand. If tomography leads to the development of analysis, liquid crystals have given rise to the first useful applications of algebraic topology - through homotopy groups - to the study of physical structures. Liquid crystals appeal also to new mathematical studies in Euclidean, Riemannian, and particularly hyperbolic geometry.

The talk given on a use of catastrophe theory leading to a positive inhibition of hemophilia was not written because of health problems suffered by the author. This is regrettable since, for the first time, the treatment of a disease until now incurable, has been made possible through the use of mathematics. However, an audio-cassette, prepared by J.P. Duport, is available from him.

To end with, I would like to suggest the study of three physical problems, all related to morphogenesis.

(1) Study experimentally and mathematically the physical morphogenesis introduced by Leduc at the beginning of the century.

Note that in the study of biological morphology involving membranes, differential geometry based on surface metrics (Cartan's metrics) should have an advantage over differential geometry based on line metrics (Riemann's metrics).

(2) Study experimentally and mathematically the trajectories of air molecules in a real balloon inflated by blowing it up. (Of course, this problem can be refined by introducing different kinds of local constraints on the shape of the balloons, and by inflating the balloon in different ways.)

(3) Study experimentally and mathematically the evolution in a convex body of sound waves created by a tiny shock on the boundary or inside the body. (The problem interests not only acousticians, but also morphogenesists; think of the problem of fecondation and very early embryology.)

I would like to thank Professors André Aragnol and André Lichnerowicz for their warm help in the preparation of the Colloquium.

The Colloquium was financially supported by the Faculté des Sciences de Marseille-Luminy, the Université d'Aix-Marseille II, the Centre National de Recherche Scientifique, and the Direction de la Coopération et des Relations Internationales of the Ministère de l'Education Nationale. The writer and all the partici-

pants address their thanks to these organizations, and to the staff of the Centre International des Rencontres Mathématiques de Luminy, who did a magnificent job of organizing our meeting.

C. P. BRUTER

EDITOR'S PREFACE

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging disciplines as "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics.

This program, Mathematics and Its Applications, is devoted to such (new) interrelations as exampla gratia:

- a central concept which plays an important role in several mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavor into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

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It used to be that physics (especially mechanics) and large parts of mathematics were inextricably intertwined. We have seen a period of separation and specialization in this respect. And now that some powerful new tools have been developed to a fine point (especially bifurcation theory, symplectic geometry and symmetry (group) ideas) they are again applied to mechanics - now enriched with problems of quantum mechanics. At the same time, these fields and a newcomer, experimental mathematics (or computer modelling), are raising fascinating mathematical questions and generating conjectures. The stated aim of the colloquium of which this collection of 19 survey papers constitutes the proceedings was "to promote the development of mathematics through applications" which is precisely one of the guiding principles of this book series.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you knows of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company they drew from each other fresh vitality anf thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange

Amsterdam, April 1983 Michiel Hazewinkel

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Wlodzimierz M. Tulczyjew

HAMILTONIAN, CANONICAL AND SYMPLECTIC FORMULATIONS OF DYNAMICS

1. INTRODUCTION

Dynamics of mechanical systems is traditionally formulated in terms of Hamiltonian vector fields [1]. Time dependent dynamics of a mechanical system is usually described by a time tependent Hamiltonian vector field or by a Poincaré-Cartan form. Recently two new formulations of dynamics have been proposed: the canonical formulation due to Lichnerowicz [3][5] and the symplectic formulation [2][6]. The canonical formulation emphasizes the importance of the Poisson structure of the phase space of a mechanical system and is related to a new approach to quantum mechanics proposed by Lichnerowicz [4]. The symplectic formulation based on the geometry of Lagrangian submanifolds of symplectic manifolds is important mainly because of its applications to relativistic dynamics [7] and because of the new interpretation of the Legendre transformation it provides [8].

The present lecture contains a review of the different formulations of time independent and time dependent dynamics of nonrelativistic mechanical systems.

2. HAMILTONIAN FORMULATION OF TIME INDEPENDENT DYNAMICS

Let (P,ω) be a symplectic manifold.

DEFINITION 2.1. A vector field

$$X:P \to TP \tag{2.1}$$

is said to be *Hamiltonian* if the form $X \lrcorner \omega$ is exact. A function

$$H:P \to \mathbb{R} \tag{2.2}$$

such that

$$X \lrcorner \omega = -dH \tag{2.3}$$

is called a *Hamiltonian* for X.

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Nonrelativistic dynamics is usually formulated in terms of Hamiltonian vector fields. The symplectic manifold (P,ω) represents the phase space of a mechanical system and trajectories of the system are solution curves of a system of ordinary differential equations represented geometrically as a Hamiltonian vector field.

3. CANONICAL FORMULATION OF TIME INDEPENDENT DYNAMICS

Let (P,ω) be a symplectic manifold and let G be the unique 2-vector field on P satisfying

$$G \llcorner (u \lrcorner \omega) = u \tag{3.1}$$

for each vector u in P.

It is known that G satisfies the *Schouten bracket* condition

$$[G,G] = 0. \tag{3.2}$$

Consequently (P,G) is a *Poisson manifold* [3][5]. The Poisson bracket $\{f,g\}$ of two functions f and g on P is defined by

$$\{f,g\} = \langle G, df \wedge dg \rangle. \tag{3.3}$$

The *Jacobi identity*

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \tag{3.4}$$

is a consequence of (3.2).

Time independent dynamics can be formulated in terms of the Poisson structure. The Hamiltonian vector field X characterized by (2.3) is defined explicitely by

$$X = -G \llcorner dH. \tag{3.5}$$

This formulation of dynamics in terms of the Poisson structure is the canonical formulation prefered by Lichnerowicz. Although equivalent to the usual Hamiltonian formulation the canonical formulation suggests a different generalization to the time dependent case.

4. LAGRANGIAN SUBMANIFOLDS

DEFINITION 4.1. A *Lagrangian submanifold* of a symplectic manifold (P,ω) is a submanifold $N \subset P$ such that $\omega|N = 0$ and $\dim N = \frac{1}{2}\dim P$.

Let M be a manifold of dimension m. Let θ_M denote the canonical 1-form defined on the cotangent bundle T^*M by

$$\langle u,\theta_M\rangle = \langle T\pi_M(u),\tau_{T^*M}(u)\rangle, \tag{4.1}$$

where u is an element of the tangent bundle TT^*M,

$$T\pi_M : TT^*M \to TM \tag{4.2}$$

is the tangent mapping of the cotangent bundle projection

$$\pi_M : T^*M \to M \tag{4.3}$$

and

$$\tau_{T^*M} : TT^*M \to T^*M \tag{4.4}$$

is the tangent bundle projection.

PROPOSITION 4.1. *The relation*

$$\mu^*\theta_M = \mu \tag{4.5}$$

holds for each 1-form $\mu : M \to T^*M$.

Proof. For each element u of TM we have

$$\begin{aligned}\langle u,\mu^*\theta_M\rangle &= \langle T\mu(u),\theta_M\rangle \\ &= \langle T\pi_M(T\mu(u)),\tau_{T^*M}(T\mu(u))\rangle \\ &= \langle u,\mu(\tau_M(u))\rangle \\ &= \langle u,\mu\rangle .\end{aligned}$$

Hence, $\mu^*\theta_M = \mu$.

As is well known $(T^*M,d\theta_M)$ is a symplectic manifold. We denote the symplectic form $d\theta_M$ by ω_M.

PROPOSITION 4.2. *Let* $F:M \to \mathbb{R}$ *be a differentiable function. The image* $N = \mathrm{im}(dF)$ *of the differential* $dF:M \to T^*M$ *is a Lagrangian submanifold of* (T^*M,ω_M).

Proof. The condition $\dim N = \frac{1}{2}\dim T^*M$ is obviously satisfied and $\omega_M|N = 0$ is equivalent to $(dF)^*\omega_M = 0$. From Proposition 4.1 we deduce

$$(dF)^*\omega_M = (dF)^*d\theta_M = d(dF)^*\theta_M = ddF = 0.$$

Hence, N is a Lagrangian submanifold.

DEFINITION 4.2. The Lagrangian submanifold $N = \text{im}\,(dF)$ is said to be *generated* by F and F is called a *generating function* of N.

The Lagrangian submanifold generated by a function F is the subset of T^*M on which the equality

$$\theta_M = d(F\circ\pi_M) \tag{4.6}$$

holds.

PROPOSITION 4.3. *Let C be a submanifold of M. The set*

$$N = \{p \in T^*M;\ x = \pi_M(p) \in C,\ \langle v,p\rangle = 0 \text{ for each } v \text{ in } T_xC\} \tag{4.7}$$

is a Lagrangian submanifold of (T^*M,ω_M).

Proof. The set N is obviously a submanifold of T^*M of dimension equal to dim M. If w is a vector tangent to N then

$$\langle w,\theta_M\rangle = \langle T\pi_M(w),\tau_{T^*M}(w)\rangle = 0$$

because $T\pi_M(w)$ is tangent to C and $\tau_{T^*M}(w)$ belongs to N. It follows that $\theta_M|N = 0$. Hence, $\omega_M|N = 0$.

5. SYMPLECTIC FORMULATION OF TIME INDEPENDENT DYNAMICS

Let $(P,\ )$ be a symplectic manifold. The mapping

$$\beta : TP \to T^*P : u \mapsto u \lrcorner\, \omega \tag{5.1}$$

is a vector bundle isomorphism. The cotangent bundle T^*P has a canonical symplectic structure independent of ω. We denote by θ_P the canonical 1-form on T^*P, and by ω_P the symplectic form $d\theta_P$. Let

$$\chi = \beta^*\theta_P \tag{5.2}$$

and

$$\rho = d\chi = \beta^*\omega_P. \tag{5.3}$$

The pair (TP,ρ) is a symplectic manifold and β is a symplectomorphism.

PROPOSITION 5.1. *The image* $\mathring{D}$ = im X *of a Hamiltonian vector field* $X:P \to TP$ *is a Lagrangian submanifold of* (TP,ρ).

Proof. Let $H:P \to \mathrm{R}$ be a Hamiltonian for X. Then $X = -G \llcorner dH = \beta^{-1}(-dH)$. Hence, $\mathring{D}$ is a Lagrangian submanifold since it is the inverse image by the symplectomorphism β of the Lagrangian submanifold of (T^*P,ω_P) generated by $-H$.

Symplectic formulations of physical theories consist in representing the constitutive equations or the dynamical equations of physical systems as Lagrangian submanifolds of suitable symplectic manifolds.

Let (P,ω) be the phase space of a mechanical system.

DEFINITION 5.1. A *dynamical system* in (P,ω) is a Lagrangian submanifold $\mathring{D}$ of the symplectic manifold (TP,ρ).

The image of a Hamiltonian vector field is a special case of a dynamical system. Other examples are encountered in relativistic dynamics and in time dependent dynamics.

6. HAMILTONIAN FORMULATION OF TIME DEPENDENT DYNAMICS [1]

Let (P,ω) be a symplectic manifold.

DEFINITION 6.1. A *time dependent vector field* on P is a mapping

$$X:P \times \mathrm{R} \to TP \tag{6.1}$$

such that for each t in R the mapping

$$X_t:P \to TP:p \mapsto X(p,t) \tag{6.2}$$

is a vector field.

DEFINITION 6.2. A time dependent vector field X is said to be *Hamiltonian* if for each t the vector field X_t is Hamiltonian. A function

$$H:P \times \mathrm{R} \to \mathrm{R} \tag{6.3}$$

such that for each t the function

$$H_t:P \to \mathrm{R}:p \mapsto H(p,t) \tag{6.4}$$

is a Hamiltonian for X_t is called a *time dependent Hamiltonian for* X.

We denote by

$$pr_1 : P \times \mathbf{R} \to P \tag{6.5}$$

and

$$t : P \times \mathbf{R} \to \mathbf{R} \tag{6.6}$$

the canonical projections. A vector field $\frac{\partial}{\partial t}$ is defined on $P \times \mathbf{R}$ by

$$\langle \frac{\partial}{\partial t}, d(f \circ pr_1) \rangle = 0 \tag{6.7}$$

for each function f on P, and

$$\langle \frac{\partial}{\partial t}, dt \rangle = 1. \tag{6.8}$$

Given a time dependent vector field X we define a field $\overline{X}$ on $P \times \mathbf{R}$ tangent to fibres of the projection pr_1 by requiring that the restriction $\overline{X}_t$ of $\overline{X}$ to the fibre over t be equal to X_t.

DEFINITION 6.3. The vector field

$$\tilde{X} : P \times \mathbf{R} \to T(P \times \mathbf{R}) \tag{6.9}$$

defined by

$$\tilde{X} = \overline{X} + \frac{\partial}{\partial t} \tag{6.10}$$

is called the *suspension* of the time dependent vector field X.

Time dependent dynamics is formulated in terms of time dependent Hamiltonian vector fields and their suspensions.

Let (P,ω) represent the phase space of a mechanical system. Let H be a time dependent Hamiltonian and X the associated time dependent Hamiltonian vector field. Trajectories of the system in the phase-time space $P \times \mathbf{R}$ are integral curves of the suspension $\tilde{X}$ of X. These trajectories are parametrized by time. They can be projected to P without any loss of information. Alternately we can disregard the parametrization and consider integral manifolds of the distribution $\tilde{D}$ on $P \times \mathbf{R}$ spanned by $\tilde{X}$.

7. THE POINCARE-CARTAN FORM

Let (P,ω) be the phase space of a mechanical system. We de-