PROBABILITY FOR ENGINEERING
with Applications to Reliability

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Preface

This book is an attempt to merge the best of two worlds. It brings precise mathematical language and terminology into an arena of modern engineering reliability probems. These include such topics as simple analysis of circuits, design of redundancy into systems, reliability of communication networks, mean time to failure, time-dependent system reliability, and uses of probability in recursive solutions to real-world problems.

All books reflect the point of view of the author. Texts on probability written by engineers usually move as quickly as possible to the applications of most interest to the writer. The mathematical foundations are presented casually and quickly, if at all, and understanding is gleaned from examples rather than from definitions and theorems. Mathematicians, on the other hand, tend to write probability books that dwell on topics such as combinatorics, labeling problems, allocation schemes, and limit theorems. Such books present little that seems relevant to the world of an undergraduate engineering student.

In teaching probability to engineering students for more than a decade, I have seen the difficulties caused by both of these kinds of textbooks. Students with good intuition are often handicapped from never having come to grips with the basic concepts. On the other hand, the importance of fundamental concepts escapes the student unless some application is in sight. For this reason, significant applications appear early in this text. Chapter 2, for example, illustrates how the idea of conditional probability can be blended with algorithmic problem solving to develop tools for reliability analysis of complex systems such as circuits, communication networks, and chemical reactors. These contemporary problems mix probability and discrete mathematics, and they require a solid understanding of the basics. But the student is rewarded with a sense of relevance that isn't matched by drawing balls out of an urn.

The essentials of an introduction to probability are found in Chapters 1, 3, 4, the first three sections of Chapter 5, and the first four sections of Chapter 6. The remaining sections of Chapter 6 introduce conditional density functions, conditional expectation, and the central limit theorem. (The De Moivre-Laplace version of the central limit theorem appears in Chapter 5.) Chapter 7 gives a brief introduction to stochastic processes, with heavy emphasis on the Poisson process. Chapter 2 applies the basic ideas of probability to reliability problems involving a variety of complex systems. Chapter 8 shows how to extend the concept of reliability to systems having components whose reliabilities vary with time.

Proper mathematical language and detail are important in formulating correct mathematical models, and this book reflects that fact. However, the book is practical rather than theoretical, and logical explanations or proofs are given only where they are an aid to intuitive understanding.

The real world confronts us with some easy problems and some hard ones. So does this book. As a result, it does not seem appropriate to treat all problems the same with regard to hints given or answers provided. This has prompted the inclusion of the section of answers, partial solutions, and hints to selected problems that appears as Appendix A. My hope is that students will give serious thought to problems *before* consulting this appendix, and that *after* consulting it they will think seriously about alternative solutions suggested or fill in missing details.

My own research in recent years has shifted into the area of reliability analysis and risk assessment. And, just as with other books, this one reflects the author's own interests. The common theme in this book is the search for reliability in an increasingly complex world. News reports on topics as diverse as arms control, strategic defense initiatives, hazardous waste dumps, and nuclear reactors all remind us of the fact that uncertainty (or lack of reliability) is central to many of the pressing issues of our day.

Lavon B. Page
September 1988

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Chapter 1: The Basics

In January 1986 the space shuttle *Challenger* exploded in midair. The space shuttle had previously been considered so safe by NASA that plans were afoot to send plutonium powered modules into orbit. The previous year an accident at the Union Carbide plant in Bhopal, India, had killed thousands of people in what was at the time the worst industrial accident in history. A few months later, the Soviet reactor at Chernobyl was to run amok and burn out of control for days, while spewing radioactivity into the atmosphere. The Soviets had thought the probability of such a major accident to be extremely low.

As long as such accidents continue, and there is every indication that they will, there will be a lot of interest in trying to determine the reliability of things like space vehicles, nuclear and chemical reactors, and such ordinary devices as automobiles, garage door openers, or heating systems. Skepticism now greets claims that something or other is "less likely to happen to you than being hit by lightning," or has "only 1 chance in 10,000 of occurring in the next 50 years." Often such claims have been based more on hope than on science.

Estimates of the reliability of equipment or complex systems depend heavily on the field of mathematics known as *probability*. Probability can be abused, just as can most tools. The best mathematical model can't produce true answers if incorrect or naive assumptions are fed into it. Even at a fairly elementary level, however, probability opens the door to the investigation of complex systems and situations. If we want to answer such questions as "What were the chances of that happening?" or "How much do we expect to gain if we make that decision?", the answer will have to be expressed in the language of probability. The purpose of this book is to present the basics of that language and to show its application to a variety of meaningful examples, with an emphasis on the idea of reliability.

Interest in probability blossomed around the gambling tables of Europe hundreds of years ago, though much earlier references can be found in Hebrew and Chinese. Many games can be analyzed by looking at the possible outcomes of an experiment, such as rolling a pair of dice or dealing some cards from a deck. Frequently something about the situation suggests that the various possible outcomes should be considered equally likely. For example, the symmetric shape of a six-sided die suggests that the six outcomes are equally probable, and the purpose in shuffling a deck of cards before dealing is to try to approximate a situation in which one arrangement of the deck is just as likely as another.

The concept of equally likely outcomes leads to a natural concept of probability. For example, since there are 13 hearts in a deck of cards, there are 13 chances out of 52 that an arbitrary card drawn from a deck will be a heart. Considering the ratio of these two numbers gives the intuitively satisfying conclusion that the probability of drawing a heart when a card is drawn from a deck should be 13/52, or 1/4.

A more skeptical person might argue that the only way to test probabilities is to experiment. For example, given a coin of unknown characteristics, the only way to determine the probability of the coin coming up heads is to toss it many times and see what happens. A mathematician might look at the situation in this way: If H_n is the number of heads obtained in the first n tosses of a sequence of tosses, the probability of heads might be taken as

$$\lim_{n \to \infty} \frac{H_n}{n}$$

Of course there's no way actually to toss a real coin an infinite number of times to evaluate the limit, but the intuitive idea is that such a limit ought to exist and should define whatever it is we mean by the probability of the coin coming up heads.

A third way that probabilities are tossed about in everyday conversation involves subjective considerations. Someone might say, for example, "Notre Dame is a 2 to 1 favorite to beat Michigan." This statement has a clear meaning as far as probabilities go. The speaker is saying that Notre Dame's chances of winning are 2 chances out of 3, which means a probability of 2/3. Such a statement is the speaker's quantitative pronouncement of his or her or somebody's opinion on the matter. Another such illustration is a weather forecaster announcing a 30% chance of rain. Presumably such a statement would be based on existing weather data and would not be purely subjective. It may be, however, that some kind of subjective guesswork went into building the weather model from which the 30% figure is obtained.

A mathematically useful treatment of probability must lay a common groundwork so that everyone is speaking the same language. Much of this groundwork consists of the elementary language of *sets* and *set operations*.

1.1 Sets and Set Operations

Intuitively, a set is simply a collection of objects. This is one of the most primitive of mathematical concepts, and thus we cannot define sets in terms of yet more elementary concepts. The common practice is to denote sets by capital letters.

Equality of two sets means that the sets consist of exactly the same elements. For example, $A = \{1, 2, 3\}$ and $B = \{2, 3, 1\}$ are equal. (There is no order associated with the elements of a set. A set is simply an unordered collection of objects.) If every element of set A is also an element of set B, then A is a *subset* of B and we write $A \subset B$. Set membership is denoted by the symbol $\in$. For example, $1 \in A$ but $4 \notin A$ in this example.

Perhaps the most useful method of defining sets is by describing the rule for set membership. For example,

$$S = \{x : x \text{ is an integer and } x > 0\}$$

is simply a way of describing S as the set of positive integers.

The three basic set operations are union, intersection, and complementation. Union and intersection are operations that are performed on collections of sets, whereas complementation is performed on a single set.

The *union* of two sets A and B is denoted by $A \cup B$ and defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

It is important to understand that "or" allows the possibility of membership in both. Thus the condition for membership in $A \cup B$ is simply that the candidate be an element of at least one of the two sets A and B.

In the definition of the *intersection* of two sets, "or" becomes "and." Thus,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Intersection thus represents the "overlap" of the two sets. If this intersection is empty, then the sets are said to be *disjoint* or *mutually exclusive*. Both the definition of union and intersection extend naturally to any collection of sets (even an infinite collection). The *union of a collection of sets* consists of all elements that belong to at least one set of the collection, whereas the *intersection of the collection* consists of the elements that all the sets have in common.

In any discussion about sets, the sets under consideration will all be subsets of some *universal set* lurking in the background. (What the universal set is should always be clear from context.) The *complement* of a set A is denoted by A^c and is defined as the set of all elements in the universal set that do *not* belong to A. For example, in the context of a discussion of the real number system, the complement of $A = \{x : x > 1\}$ is

$$A^c = \{x : x \leq 1\}$$

The *difference* of two sets, $A - B$, is defined as $A \cap B^c$.

Some of the important elementary laws governing the set operations are as follows:

Associative law for union	$A \cup (B \cup C) = (A \cup B) \cup C$
Associative law for intersection	$A \cap (B \cap C) = (A \cap B) \cap C$
Commutative law for union	$A \cup B = B \cup A$
Commutative law for intersection	$A \cap B = B \cap A$
Distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
De Morgan's laws	$(A \cup B)^c = A^c \cap B^c$
	$(A \cap B)^c = A^c \cup B^c$

A simple way to verify some elementary set identities is to use *Venn diagrams*. The idea is to visualize a set as being represented by a region in the plane. Figure 1.1 illustrates the use of this concept with regard to the first distributive law above. The shaded area in the figure is the region corresponding to the sets on either side of the equation in the first distributive law. One way to check equality of sets is to construct a Venn diagram for each set and to observe that the Venn diagrams coincide.

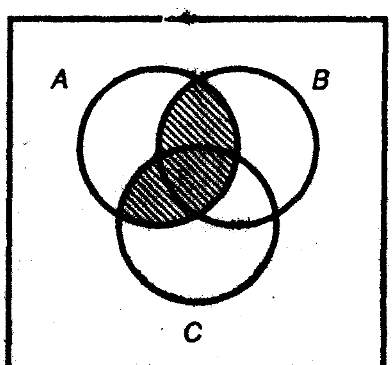
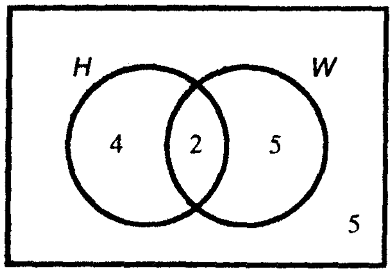


Figure 1.1 Venn diagram representing the set $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Example 1.1. A group of 9 men and 7 women are administered a test for high blood pressure. Among the men, 4 are found to have high blood pressure, whereas 2 of the women have high blood pressure. Use a Venn diagram to illustrate this data.

Solution:

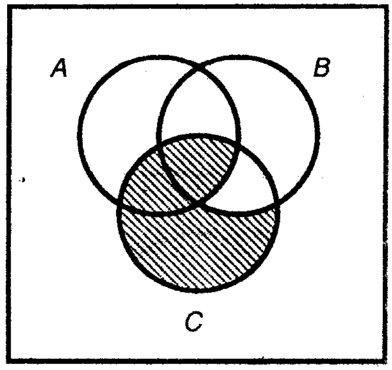


The circle labeled H represents the 6 people having high blood pressure, and the circle labeled W represents the 7 women. The numbers placed in the various regions indicate how many people there are in the category corresponding to the region. For example, there are 4 people who have high blood pressure and who are not women. Such people are in set H but not in set W; that is, they belong to the set $H \cap W^c$. The number 5 in the lower right corner indicates the number of men without high blood pressure.

The decision to use circles to represent "high blood pressure" and "women" was quite arbitrary. We could just as well use circles for "low blood pressure" and "men." (See Problem 1.25.)

Example 1.2. If A, B, and C are sets, draw a Venn diagram and shade the region corresponding to the set $(A \cup B^c) \cap C$.

Solution: The best way to arrive at the following figure is in a step-by-step manner. First, decide what region corresponds to the set $A \cup B^c$. This will include all the space *inside* the A circle and all the space *outside* the B circle. Then, the final step is to observe what part of this region lies *inside* of C (since the final operation is intersection). If you like you can think of spray painting everything inside the A circle, then spray painting everything outside the B circle, and then looking to see what part of the inside of circle C is painted. The region that represents the set $(A \cup B^c) \cap C$ is shown in the figure.



Example 1.3. If A, B, and C are the following sets of characters, then determine the set $(A \cup B^c) \cap C$. Here we will consider the universal set to be *all* 26 letters of the alphabet.

$$A = \{a, h, t\}$$
$$B = \{d, g, b, t\}$$
$$C = \{d, a, g\}$$

Solution: While it is possible to represent this information in a Venn diagram, it certainly isn't necessary. Simply observe that B^c consists of all letters except d, g, b, and t. Therefore, $A \cup B^c$ consists of all letters except d, g, and b. So the only letter that $A \cup B^c$ has in common with C is a. Conclusion: $(A \cup B^c) \cap C = \{a\}$. Notice that Problem 1.27 asks you to observe this in the context of a Venn diagram.

1.2 The Sample Space

The *sample space* is roughly "the set of all possible observations or outcomes" of whatever is under discussion. This idea is best illustrated through examples.

Example 1.4. If a coin is tossed, the sample space could be taken to be the set $S = \{\text{H, T}\}$. If an ordinary six-sided die is rolled, the sample space could be taken to be the set $S = \{1, 2, 3, 4, 5, 6\}$.

Example 1.5. A card is drawn from a standard deck of 52. Here one could take the sample space S to be

$$S = \{2\clubsuit, 2\diamondsuit, 2\heartsuit, 2\spadesuit, 3\clubsuit, 3\diamondsuit, \cdots, K\spadesuit, A\clubsuit, A\diamondsuit, A\heartsuit, A\spadesuit\}$$

A simpler convention would be to agree to think of the cards as being identified with the numbers 1,2, $\cdots$, 52 and to simply think of S as consisting of these 52 numbers. It is important to realize that the particular bookkeeping scheme used is not very important compared to the conceptual understanding of what kind of set is an appropriate model. Whatever notation is used here, the sample space is a set of 52 elements.

Example 1.6. Suppose a simple electric circuit has two components, say A and B. Either component can be "good" or "bad" in the sense that the component may or may not be in working order. If we are interested in all possible states of the circuit, the sample space used could be $S = \{GG, GB, BG, BB\}$, where the convention might be that "GB," for example, means that component A is good and component B is bad.

Example 1.7. A pair of dice is rolled. Let's refer to them as "red" and "green." An appropriate choice for the sample space for this experiment would be the set $S = \{(1, 1), (1, 2), (1, 3), \cdots, (6, 5), (6, 6)\}$, where, for instance, we might agree that (3, 5) represents the outcome of 3 on red and 5 on green. The set S contains 36 elements since either die can come up 6 different ways and 6×6 is 36. (There's a basic underlying principle here that some elementary texts call the *multiplication principle*. When one task can be performed in m different ways and another task can be performed in n different ways, then the number of different ways of performing the two operations together is mn.)

In playing many games, one is not interested in the individual numbers that appear on the dice, but rather in the sum. In this case it might be tempting to take the sample space to be $S = \{2, 3, 4, \cdots, 11, 12\}$. This is not necessarily wrong, but it does sacrifice information. For example, if this sample space is used, one can no longer answer questions such as "Did the red die show an even number?" The result on the red die is not even being recorded. Another reason for caution is that the outcomes of this set of possible sums are not equally likely. For example, a sum of 2 occurs only if both dice show the number 1, whereas a sum of 7 occurs in six different ways. It is often necessary to consider sample spaces in which individual outcomes don't all have the same probability. One needs, however, to be aware when this is the situation. A frequent naive mistake is to assume that outcomes are equally likely when they are not. It is common to refer to a sample space in which all the elements are considered equally probable as a ***uniform*** *sample space.*

Example 1.8. When binary data is transmitted, the output can be thought of as a string of 0's and 1's. (In electrical transmission, a voltage above a certain level could be defined as 1 and below a certain level as 0.) If 4 bits are transmitted, what would be an appropriate sample space to represent the possibilities?

Solution: The logical choice would be to take the sample space to be all ordered quadruples of binary digits. In other words, $S = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$. These 16 outcomes are simply the numbers from 0 to 15 written in binary form. There are 16 elements of the sample space because $2^4 = 16$. Eight bits (commonly referred to as a byte) can represent $2^8 = 256$ different possibilities. This is equivalent to saying that if 8 bits are transmitted, then the sample space for all possible outcomes has 256 elements.

1.3 Basic Properties of Probabilities

The word *event* is commonly used in everyday language, and often we speak of a particular event "occurring." In probability discussions, you should think of an *event* as a subset of the sample space. For example, if we say when a die is rolled that "an even number occurs," we are saying that the observed outcome lies in the set $E = \{2, 4, 6\}$. We are describing E verbally by saying E is the event "that an even number occurs," but this is simply an alternate way of saying that E is the given set of outcomes. The reason for talking about events occurring is that this language is so common and useful in everyday speech. In talking about probability, to say that an event occurs is simply to say that the observed outcome is one of the elements of the event. This mathematical language is consistent with common speech. You should keep in mind, though, that in the language of probability there is a precise mathematical meaning to such expressions.

There are three basic axioms that characterize what is meant by a ***probability measure***. The notation $P(A)$ represents the probability that the event A occurs, and the assumption is that probabilities always behave according to the these rules.

The Basic Axioms for a Probability Measure

1. For each event A, $P(A) \geq 0$.
2. $P(S) = 1$, where S denotes the sample space.
3. If A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$.

The intuitive basis for all three axioms should be obvious. When we say that something has probability 1/10, we mean that there is one chance in 10 that it will occur. A negative probability has no conceivable meaning. Similarly, a probability of 1 represents absolute certainty, and probabilities greater than 1 would be meaningless. The last axiom is very important in that it allows the probability of events (in the case of a finite sample space) to be computed in terms of the probabilities of the individual elements that make up the events. This will be demonstrated shortly. (See Equation 1.1.)

Additional Properties of Probabilities

1. $P(\varnothing) = 0$, where $\varnothing$ denotes the empty set.
2. $P(A) \leq P(B)$ if $A \subset B$, and $P(A) \leq 1$ always.
3. $P(A - B) = P(A) - P(A \cap B)$.
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ always.
5. If $A_1, A_2, \cdots, A_n$ are disjoint events (no two events having any elements in common), then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n)$$

Property 1 is an immediate consequence of Axiom 3. (Simply take A and B both to be the empty set in Axiom 3, and you have $P(\varnothing) = 2\,P(\varnothing)$, which implies $P(\varnothing) = 0$.)

If $A \subset B$ then $B = A \cup (B - A)$, and this is a disjoint union. Thus, from Axiom 3 we know that $P(B) = P(A) + P(B - A)$. The right side here is greater than or equal to $P(A)$ because $P(B - A) \geq 0$ (Axiom 1), and this proves Property 2.

Property 3 is a result of the fact that $A = (A \cap B) \cup (A - B)$, and the union here is disjoint.

Property 4 may be obtained by first noting that

$$P(A \cup B) = P(A) + P(B - A)$$

This is a special case of Axiom 3. From Property 3, however, we know that

$$P(B - A) = P(B) - P(A \cap B)$$

The truth of Property 5 is a consequence of mathematical induction. In fact, if

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + \cdots + P(A_k)$$

then it follows that

$$P(A_1 \cup A_2 \cup \cdots \cup A_{k+1}) = P(A_1) + \cdots + P(A_{k+1})$$

simply by letting $A_1 \cup A_2 \cup \cdots \cup A_k$ and A_{k+1} play the roles of A and B in Axiom 3.

For finite sample spaces it is always possible to compute the probability of an event by focusing on the individual elements that make up the event. For if the sample space is $S = \{s_1, s_2, \cdots, s_n\}$, then, according to Property 5, the probability of any event $A \subset S$ may be computed via

$$P(A) = \sum_{s_k \in A} P(\{s_k\}) \tag{1.1}$$

In other words, the probability of any (finite) event may be computed by simply adding up the probabilities of the individual elements of the event.

Later in this book we will see that probabilities of events often must be approached from a different perspective when the sample space is infinite. The idea of computing probabilities of all events via a sum, as in Equation 1.1, must be abandoned. In Chapter 5 we will see that in "continuous" models, integration takes the place of summation.

Continuous models arise, for instance, when measurements are being made on some kind of continuous scale and one wishes to think of an interval of possible values. For example, think of the experiment of "selecting a random number between 0 and 1." Clearly it would be intuitively satisfying to think that the probability is 1/2 that the number should come from the subinterval [0, 1/2], or 1/5 that the number chosen should be in the interval [3/5, 4/5]. In fact, we would like to know that the probability of the number falling in any particular subinterval is simply the length of the subinterval. Simple as this situation sounds, the actual demonstration that there is a probability measure on the interval [0, 1] that has these