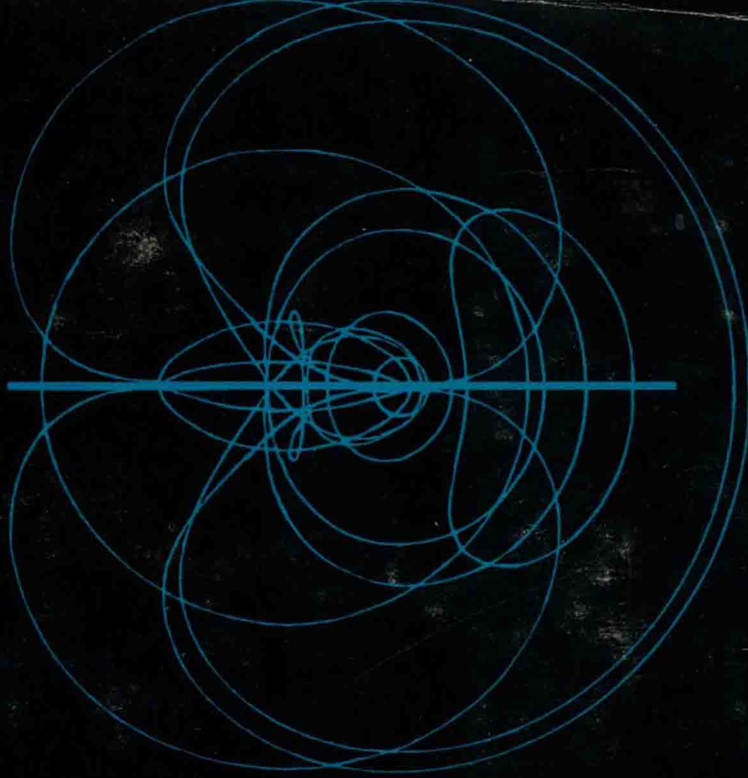

THEORY OF ORBITS

The Restricted Problem of Three Bodies

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The Restricted Problem of Three Bodies

Foreword

The subject of this treatise, the restricted problem of three bodies, occupies a central place in analytical dynamics, celestial mechanics, and space dynamics. Entry into celestial mechanics and space dynamics can be gained by the study of the problem of two bodies. To penetrate the fundamental problems, the number of participating bodies must be increased from two to three. This step is critical. Not only is the two-body problem solved—and the meaning of "solution" may be different for astronomers, engineers, and mathematicians—but a general understanding exists regarding this dynamical system. The problem of three bodies on the other hand is neither solved nor is the behavior of the dynamical system completely understood.

The solar system provides few applications of the general problem of three bodies. This results in an unusual situation where a more general problem having considerable complexity is less useful than a comparatively simple formulation. Also it is important to realize that more is known about the restricted problem than about the general problem.

This volume is strongly influenced by the creators of modern dynamics, H. Poincaré and G. D. Birkhoff. Poincaré, in his *Méthodes Nouvelles de la Mécanique Céleste* and also in his famous *Mémoire Couronné*, "Sur le problème des trois corps et les équations de la dynamique," uses the problem of three bodies as his favorite example when presenting his work in dynamics. The same is true for G. D. Birkhoff's *Dynamical Systems*, and for C. L. Siegel's *Vorlesungen über Himmelsmechanik*. A. Wintner's *Analytical Foundations of Celestial Mechanics* was originally planned to treat the problem of three bodies, especially the restricted problem, but it actually presented more of the mathematical foundations than of the celestial mechanics. It is interesting to note that H. Happel's book is entitled *Das Dreikörperproblem*, and the subtitle reads "Vorlesungen über Himmelsmechanik," while the second volume of K. Stumpff's *Himmelsmechanik* displays the subtitle "Das Dreikörperproblem." How intimately the problem of three bodies is connected with celestial mechanics and with dynamics in general when titles, subtitles, contents, applications, and examples become interchangeable!

The applications of the restricted problem to celestial mechanics form the basis of some lunar and planetary theories. The modern applications to space mechanics are probably even more cogent if not more numerous than the classical applications. The implications of the restricted problem for cosmogony and stellar dynamics are also numerous. Finally, it can be shown that a great variety of dynamical systems can be presented by equations of motion which are formally identical with the equations of the restricted problem. One measure of the importance of a scientific endeavor is its effect on peripheral fields. While authors from Euler to Siegel recognized astronomy and dynamics as the only peripheral fields, today we know that space mechanics and stellar dynamics are fields which benefit equally.

The interest in space sciences rejuvenated celestial mechanics, and the well-established tools of the latter were immediately applied. Some of the problems were not really new and the proven methods of classical celestial mechanics—in the hands of the masters—produced immediate results. I think of several solutions of the drag-free earth-satellite problem, for instance, which today may be considered settled. It is a perturbation of the two-body problem, and the success in solving it is partly explained by the popularity of satellite problems in classical celestial mechanics. Other problems in space dynamics, closely associated with the restricted problem, are of considerable importance and interest today. Many of these problems are new, and in what follows one of them will be contrasted to a classical problem. Consider the famous classical three-body problem, the sun-earth-moon combination and the determination of the motion of the moon. We might think about two large bodies, the sun and the earth, which move around each other in approximate circles, and in their field a third body, the moon, which moves on an approximate ellipse. This configuration is stationary in a sense, since no collisions take place. This is also true for the motion of a Trojan asteroid under the continued influence of the sun and Jupiter. On the other hand, one of the central problems in space science is to create artificial bodies which may be required to move on orbits connecting the close neighborhood of two natural celestial bodies. Sometimes collision orbits are desired. Problems with close approaches and collisions were hardly ever treated in classical celestial mechanics and these problems became important in the new science of space dynamics.

The use of three essentially different approaches to dynamics, the qualitative, the quantitative, and the formalistic, is dictated by the special advantages of each and is described in the Introduction, where a number of references to the history of the restricted problem are also given.

The first chapter introduces the problem of three bodies and formulates the equations of motion in inertial and in rotating coordinate systems. The relation of the restricted problem to the general problem of three bodies is described and illustrated with examples. Several applications to cosmogony and stellar dynamics are also outlined. Chapter 2 discusses reductions of the problem and offers a comprehensive treatment of streamline analogies.

Chapter 3 is concerned with regularization and shows how the equations of motion can be written in a system free of singularities. This subject is the feature which distinguishes a work on classical celestial mechanics from one on modern applications. This chapter is probably the most important one for the reader who is working in the field of space mechanics. Chapter 4 is devoted to the principal qualitative aspect of the restricted problem—the curves of zero velocity, several uses of which are discussed. The regions of permissible motion and the location and properties of the libration points are established. Motion and nonlinear stability in the neighborhood of these equilibrium points are treated in detail in Chapter 5.

Chapter 6 contains a short introductory treatment of Hamiltonian dynamics in the extended phase space. Chapter 7 applies the principles and methods of the previous chapter to the restricted problem and to its regularization. The generating functions that are used are derived with emphasis on justification and motivation. A natural way to introduce the concept of perturbation theory is presented.

Chapter 8 discusses the problem of two bodies in a rotating coordinate system and treats periodic orbits in the restricted problem, following H. Poincaré and G. D. Birkhoff. Chapter 9 presents the quantitative aspects of the restricted problem. The results of G. Darwin, E. Strömgren, and F. R. Moulton are discussed and several of the recently established lunar and interplanetary orbits in the Soviet and American literature are compared. Chapter 10 is devoted to modifications of the restricted problem, such as the elliptic problem, the three-dimensional problem, and Hill's problem.

V. SZEBEHELY

Preface

This volume has been developed from my lectures and seminars on various aspects of celestial mechanics, dynamics, the restricted problem of three bodies, periodic orbits, regularization, and space dynamics. While directed primarily to the graduate student, it is intended to be sufficiently comprehensive to serve as a reference and advanced text on many applications of celestial mechanics. One purpose is to familiarize those readers who are concerned with the space applications of celestial mechanics with the next step after the problem of two bodies. The student of celestial mechanics will find both classical studies and recent developments in the restricted problem of three bodies with a survey of the pertinent literature.

This is the first book devoted to the theory of orbits in the restricted problem. My aim is to build a bridge between books written for the astronomer, mathematician, space engineer, and student of dynamics. Instead of developing the subject separately for each of these professions, it is hoped that the single subject of this volume will be useful for all its readers. Astronomers will find more references to analytical dynamics than is usual in textbooks on celestial mechanics; workers in the field of dynamics will read about astronomical applications; the needs of mathematicians and engineers will be met by the problem of establishing the totality of possible motions of our dynamical system.

Teaching experience shows that students are interested in historical reviews and remarks in the field of celestial mechanics, which is so rich in traditions and in cultural background material. Such comments are collected at the end of each chapter with the discussions of the pertinent references. Most chapters contain a generous amount of basic mathematical information. I make it a point to extend the foundations more than necessary for the building, in order to establish a more solid edifice and offer to the reader the opportunity of proceeding with his own applications.

My guiding principle has been to inform the reader of the motivation and purpose of the developments, hoping to inspire his enthusiastic interest in the subject. I try to avoid unnecessary *epsilontics* in the mathematical parts and highly specialized and undefined terms in the applications. Mathematics is a tool in dynamics, not a goal. The Wintnerian turnaround from the problem of three bodies to mathematics is avoided, and an attempt is made to emphasize the dynamics. I subject the brilliance of Poincaré and of G. D. Birkhoff to scrutiny and explanation rather than to competition. My aims are to summarize G. Darwin's eloquence, to expand Siegel's terseness, to generalize Charlier, and to particularize Moulton and E. Strömgren. Special attention is paid to the Soviet literature of the past two or three decades; it contains many significant contributions to celestial mechanics and space dynamics. Recent numerical results on earth-moon trajectories are compared with previous results, and classical orbit computations are brought up to date.

April, 1967 V. SZEBEHELY

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I gratefully acknowledge the cooperation and assistance received from my colleagues and students. The former read parts of my manuscript and helped to enrich the text by their illiminating discussions. I am especially indebted to Dr. R. Arenstorf, Dr. A. Deprit, Dr. B. Garfinkel, Dr. C. Lanczos, and Dr. J. Message for their comments. Dr. G. M. Clemence has read the complete manuscript; Drs. D. Brouwer and G. Giacaglia large parts of it. They made many valuable suggestions.

I extend sincere thanks to my graduate students. The computations involved in preparing the major tables were performed by Miss C. Williams. Several of the complicated figures were drawn by an automatic plotter, imputs to which were prepared by Dr. D. Pierce and Mr. M. Standish. Several of the orbits were computed by Messrs. S. Knowles and J. Lieske. In these undertakings the facilities of the Yale Computer Center were used. The burden of reading the proofs was shared mostly by Messrs. D. Bettis, J. Breedlove, P. Esposito, R. Laubscher, and P. Nacozy, graduate students. The tables were put in their final form by Dr. and Mrs. R. Duncombe of the U.S. Naval Observatory.

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Contents

Introduction

The purpose of dynamics is to characterize the totality of possible motions of a given dynamical system. Such a characterization does not necessarily mean an explicit, closed-form, general solution of the problem since this is seldom possible, and when it is possible, it is most of the time neither meaningful nor helpful in understanding the behavior of the system. An example is the problem of two bodies, which is considered solved since the properties of the totality of possible motions are known. Although the coordinates describing the motion of the bodies participating in the problem cannot be represented as explicit functions of the time in closed form, the problem is nevertheless considered solved.

Qualitative, quantitative, and formalistic dynamics are the three major approaches to the understanding of the behavior of dynamical systems. The qualitative approach is probably the most elegant and sometimes the most powerful. The formalistic method is the basis of classical celestial mechanics. The quantitative approach is often the most popular among astronomers and engineers who may want to find one particular solution of a problem rather than to study the behavior of the dynamical system. Examples are the ephemerides of the planets, representing particular solutions of the astronomers' n-body problem and Apollo trajectories, being particular solutions of the engineers' problem.

Qualitative methods in dynamics are well suited to the treatment of such questions as stability, existence problems, integrability, and reducibility. The names of H. Poincaré and G. D. Birkhoff are associated with qualitative dynamics; Hill's name is seldom thought of in this connection, in spite of his use of the zero velocity curves to establish limiting regions. His method is probably one of the most powerful and successful qualitative ideas in the restricted problem.

It is sometimes said of qualitative dynamics that its results are not helpful to "practical" men (to the "users" as opposed to the "creators"). This misconception is partly because some of the qualitative results in dynamics have not yet been interpreted and some of these results are of theoretical interest only.

Knowledge of certain qualitative properties of a dynamical system may be much more valuable than numerical solutions. An example is the existence question of periodic orbits. Solutions of nonintegrable dynamical systems are never known along the whole time axis unless they are of periodic or asymptotic nature. This is seen when we consider an attempt to establish a particular solution of the differential equations of a dynamical system with an electronic computer. Not attempting for the moment to evaluate such an undertaking, let us visualize the computer output as the time increases without limit and as various error sources contribute to the printouts. Unless some systematic behavior of the result is discovered, sooner or later the computer output becomes meaningless and no valuable information about the dynamical system will be obtained along the whole time axis. The orbit or the behavior of the system will remain unknown in spite of the numerical work.

Another example is furnished by one of the fundamental questions of dynamics: the description of the totality of possible motions of a dynamical system. For nonintegrable systems this is a major problem as no closed-form general solution is available. The practical importance of knowing all possible orbits between the earth and the moon does not need emphasis, since selection of an orbit "best" suited for a certain mission requires information regarding the possible choices. A formalistic approach to this problem is not fruitful, for even if it should furnish convergent series which give the general solution, the nature, the properties, and the totality of the solution could not in general be determined from such series. The quantitative approach to this problem is to select a region of the initial conditions which is of practical interest and to compute as many orbits as possible in this region. This set of orbits is called the "totality of orbits of interest." The deficiency in this approach is the possible omission of useful orbits or of whole families of useful orbits. When the possible range of initial conditions is considerable, the establishment of families of orbits according to six varying initial conditions is almost a hopeless task numerically. The description of the totality of possible motions should come from a combined approach (numerical and formalistic) with qualitative dynamics leading and organizing the steps. One of the most practical and most important problems in applied celestial mechanics, the selection of a suitable orbit, is therefore equivalent to one of the most advanced problems of qualitative dynamics.

Turning now to the formalistic approach we enter the stronghold of classical celestial mechanics. The formalistic methods are also called general perturbation methods, and the principal mathematical tools are series expansions. In order to have a general perturbation method the initial conditions are kept arbitrary in the solution. Justification of the method from a mathematical point of view requires scrutiny of the convergence of the series with respect to the variables. It is ironic that one of the qualitative results of dynamics, attributed to Poincaré, states that the series used in celestial mechanics are in general divergent. Nevertheless, finite parts of such series are often extremely useful in celestial mechanics since they do give results in agreement with observations. Questions connected with the behavior of the system as the time increases to infinity cannot, of course, be answered by such series solutions. The classical series of celestial mechanics become of little use when bodies approach each other closely and when they collide. Since such orbits are of central importance in modern dynamics, new formalistic approaches have had to be devised.

The hopefully expected ultimate answer of representing the totality of solutions as "simple" functions of the initial conditions and of time may come from formalistic dynamics. Such a result can probably be expected from a combined effort of the three major approaches with the formalistic approach taking the lead. Newton's approach to dynamics was to find just such explicit expressions representing the motion of dynamical systems. Advances in celestial mechanics and in other branches of science with mathematical orientation show more or less the same steps. First comes the attempt to describe the field of interest with simple analytic expressions. This leads of necessity to successive approximations and series solutions if the first attempt for simple closed-form solutions fails. Those fields, such as the "solvable" dynamical systems, in which the first step furnishes results, are considered solved and are soon abandoned. The quantitative approach to dynamics is not unlike the first step because it gives a particular solution in a simple form: a set of numbers representing the coordinates as functions of time. Those fields in which sufficient interest exists for establishing general solutions, but which at the same time are not "integrable" and therefore are not amenable to simple general solutions, are graduated to the second phase of mathematical physics: to series solutions. Some problems are solved at this stage if the series solutions furnish the properties of the general solution. This is seldom the case