Planned Textbook For University

Advanced Mathematics

(II)

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Mingming Chen
Zhenyu Guo Jingxian Yu Jinqiu Li

Chemical Industry Press

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· Beijing ·

The aim of this book is to meet the requirement of bilingual teaching of advanced mathematics. The selection of the contents is in accordance with the fundamental requirements of teaching issued by the Ministry of Education of China. And base on the property of our university, we select some examples about petrochemical industry. These examples may help readers to understand the application of advanced mathematics in petrochemical industry.

This book is divided into two volumes. The first volume contains calculus of functions of a single variable and differential equation. The second volume contains vector algebra and analytic geometry in space, multivariable calculus and infinite series.

This book may be used as a textbook for undergraduate students in the science and engineering schools whose majors are not mathematics, and may also be suitable to the readers at the same level.

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Preface

English is the most important language in international academia. In order to strengthen academic exchange with western countries, many universities in China pay more and more attention to the bilingual teaching in classrooms in recent years. Considering the importance of advanced mathematics and scarcity of bilingual mathematics textbook, we have written this book.

The main subject of this book is calculus. Besides, it also includes differential equation, analytic geometry in space, vector algebra and infinite series. This book is divided into two volumes. The first volume contains calculus of functions of a single variable and differential equation. The second volume contains vector algebra and analytic geometry in space, multivariable calculus and infinite series.

We have attempted to give this book the following characteristics:

① The content of this book is based on the Chinese textbook "advanced mathematics (sixth edition)" which is written by department of mathematics of Tongji University. The readers may read this book and use the Chinese textbook "advanced mathematics" as a reference. It may help readers to understand the mathematical contents and to improve the level of their English.

② In order to train the mathematical idea and ability of the students, we use some modern idea, language and methods of mathematics. We also bring in some mathematical symbol and logical symbol.

③ We pay more attention to the application of mathematics in practical problems. We have added some other examples and exercises in physics, chemistry, economics and even daily life.

④ Considering the different teaching requirements in different schools, we mark some difficult sections and exercises by the symbol "*". Teachers and students may choose suitable contents as required.

In this volume, Chapter 8 is written by Jinqiu Li, the first four sections in Chapter 9 are written by Min Liu, the rest five sections in Chapter 9 are written by Zhenyu Guo, Chapter 10 is written by Jingxian Yu, Chapter 11 is written by Xiaoying Zhao, Chapter 12 is written by Mingming Chen. All the chapters are checked and revised by Mingming Chen.

We hope this book can bring readers some help in the studying and teaching of bilingual mathematics. Due to the limit of our ability, it is impossible to avoid some errors and unclear explanations. We would appreciate any constructive criticisms and corrections from readers.

Authors
Liaoning Shihua University
2010-11

Contents

Chapter 8

Vector Algebra and Analytic Geometry of Space

In analytic geometry of plane, points of plane and ordered pair, graphs of plane and equations are matched by method of coordinate. Then, problems of geometry are studied with algebraic method. Analytic geometry of space is established according to the similar method.

This chapter introduces the concepts of vector algebra and analytic geometry of space. These concepts are very important not only for studying calculus of functions of several variables in next chapter, but also for applications in physics, mechanics, other sciences, and engineering.

8.1 Vectors and their linear operations

8.1.1 The concept of vector

Some of the things we measure are determined simply by their magnitudes. To record mass, length or time, for example, we need only write down a number and name an appropriate unit of measure. We need more information to describe a force, displacement, or velocity. To describe a force, we need to record the direction in which it acts as well as how large it is. To describe a body's displacement, we have to say in what direction it moved as well as how far. To describe a body's velocity, we have to know where the body is headed as well as how fast it is going.

A quantity that has both magnitude and direction, such as force, displacement, or velocity, is called **a vector**. A vector is usually represented by a line segment with an arrow, a **directed line segment**. The length of the directed line segment represents the magnitude of the vector and the arrow points in the direction of the vector. The vector represented by the directed line segment from the initial point A to the terminal point B is denoted by $\overrightarrow{AB}$ (Figure 8-1). In

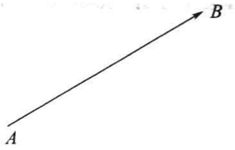

Figure 8-1

textbooks, vectors are usually written in boldface letters, such as **a**, **r**, **v** and **F**. In handwritten form, it is customary to draw small arrows above the letters, such as $\vec{a}$, $\vec{r}$, $\vec{v}$ and $\vec{F}$.

The magnitude of a vector is called the **length** of the vector. The length of the vectors $\overrightarrow{AB}$, **a** and $\vec{a}$ are denoted by $|\overrightarrow{AB}|$, $|\mathbf{a}|$ and $|\vec{a}|$. A vector whose length is 1 is called a **unit vector**. A unit vector whose direction is the same as that of **a** is denoted by $\mathbf{e}_a$. A vector whose length is 0 is called the **zero vector** and is denoted by **0** or $\vec{0}$. The initial point of the zero vector coincides with its terminal point. It is the only vector with no specific direction.

It is seen from the definition of vector that a vector is determined completely by its magnitude and direction and is independent of the location of its initial point and terminal point. Therefore, two vectors **a** and **b** are said to be equal if they have the same length and direction, denoted by $\mathbf{a}=\mathbf{b}$.

A vector is called the **negative** of **a**, if it has the same length as **a** but points in the opposite direction, denoted by $-\mathbf{a}$. Obviously, we have $\overrightarrow{AB}=-\overrightarrow{BA}$.

Let **a** and **b** be two nonzero vectors. Then **a** and **b** are said to be **parallel** or **collinear**, if their directions are the same or opposite, denoted by $\mathbf{a}//\mathbf{b}$. The vectors **a** and **b** are said to be **orthogonal** or **perpendicular**, if the directions of **a** and **b** are orthogonal, denoted by $\mathbf{a}\perp\mathbf{b}$.

Suppose that $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k$ $(k\geqslant 3)$ are k vectors with a common initial point. If they lie in the same plane, we say that these vectors are **coplanar**. It is easy to see that any two vectors are coplanar.

Let **a** and **b** be two nonzero vectors. Select a point O of space arbitrarily. Make $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. The angle between the two vectors **a** and **b** is $\angle AOB$ which is no more than π $(0\leqslant \angle AOB\leqslant\pi)$, denoted by $(\widehat{\mathbf{a},\mathbf{b}})$ or $(\widehat{\mathbf{b},\mathbf{a}})$ (Figure 8-2). If $\mathbf{a}=\mathbf{0}$ or $\mathbf{b}=\mathbf{0}$, $(\widehat{\mathbf{a},\mathbf{b}})$ can be any value between 0 and π.

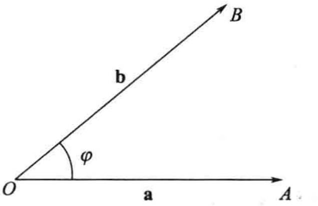

Figure 8-2

8.1.2 Vector linear operations

8.1.2.1 Vector addition

Suppose a particle moves from A to B, so its displacement vector is $\overrightarrow{AB}$. Then the particle changes direction and moves from B to C, with displacement vector $\overrightarrow{BC}$. The combined effect of these displacements is that the particle has moved from A to C. The resulting displacement vector $\overrightarrow{AC}$ is called the sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and we denote $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$.

In general, if we start with vectors **a** and **b**, we first move **b** so that its tail coincides with the tip of **a** and define the sum of **a** and **b** as follows.

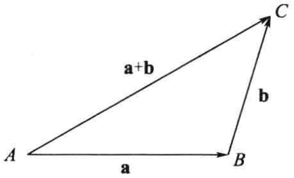

Figure 8-3

Definition 1 (Triangle law of vector addition) Suppose that **a** and **b** are any two vectors and A is any point. Make $\overrightarrow{AB}=\mathbf{a}$. Draw vector $\overrightarrow{BC}=\mathbf{b}$ starting at the terminal point B of **a**. Connect A and C (Figure 8-3), then the vector $\overrightarrow{AC}=\mathbf{c}$ is called the **sum** of **a** and **b**, denoted by $\mathbf{a}+\mathbf{b}$. That is $\mathbf{c}=\mathbf{a}+\mathbf{b}$.

If the vectors **a** and **b** are not parallel, we can also find their sum according to the following **parallelogram law.** We take an arbitrary point A, draw $\overrightarrow{AB}=\mathbf{a}$, $\overrightarrow{AD}=\mathbf{b}$, and take AB and AD as the adjoining sides of a parallelogram $ABCD$, connect diagonal AC (Figure 8-4), then $\mathbf{a}+\mathbf{b}=\overrightarrow{AC}$.

The vector addition satisfies the following laws.

① Commutative law $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$.

② Associative law $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$.

Here, ① and ② are illustrated geometrically in Figure 8-4 and Figure 8-5 respectively.

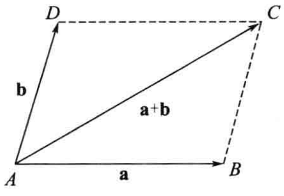

Figure 8-4

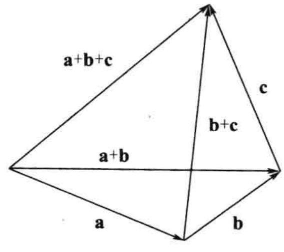

Figure 8-5

As shown in Figure 8-4,

$$\mathbf{a}+\mathbf{b}=\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}=\mathbf{c}$$

$$\mathbf{b}+\mathbf{a}=\overrightarrow{AD}+\overrightarrow{DC}=\overrightarrow{AC}=\mathbf{c}$$

so $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$.

As shown in Figure 8-5, make the sum of $\mathbf{a}+\mathbf{b}$ and $\mathbf{c}$, then make the sum of $\mathbf{a}$ and $\mathbf{b}+\mathbf{c}$. We find the same result, that is $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$.

According to the triangle law of vector addition, we have the sum of n vectors. Let the terminal point of one vector be the initial point of next vector. Make vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ ($n \geqslant 3$) one after the other. Then make the vector start at the initial point of the first vector and end at the terminal point of the final vector, we have the sum of n vectors, that is

$$\mathbf{a}_1+\mathbf{a}_2+\cdots+\mathbf{a}_n$$

See Figure 8-6, $\mathbf{s}=\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4+\mathbf{a}_5$.

Moreover, we define the **difference** of two vectors $\mathbf{b}$ and $\mathbf{a}$ by

$$\mathbf{b}-\mathbf{a}=\mathbf{b}+(-\mathbf{a})$$

that is the sum of $\mathbf{b}$ and $-\mathbf{a}$ (Figure 8-7(a)).

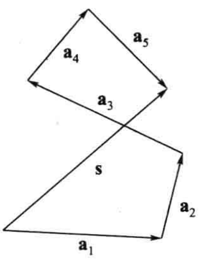

Figure 8-6

Specially, when $\mathbf{b}=\mathbf{a}$, we get

$$\mathbf{a}-\mathbf{a}=\mathbf{a}+(-\mathbf{a})=\mathbf{0}$$

Suppose that $\overrightarrow{AB}$ is an arbitrary vector and O is an arbitrary point. Obviously, we have

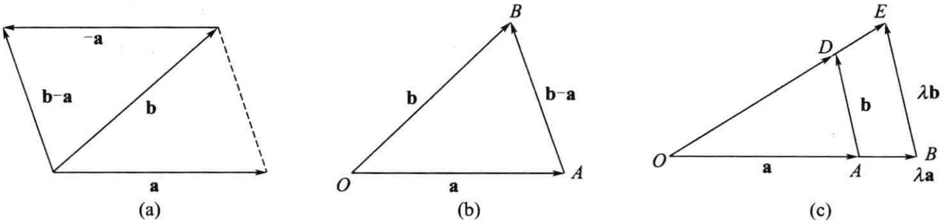

Figure 8-7

$$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}=\overrightarrow{OB}-\overrightarrow{OA}$$

Therefore, according to the triangle law of vector addition, if the initial points of **a** and **b** are the same, then the vector starting at the terminal point of **a** and ending at the terminal point of **b** is just the difference of **b** and **a**, that is **b**−**a** (Figure 8-7(b)).

8.1.2.2 Scalar multiplication

It is possible to multiply a vector by a real number. (In this context we call the real number a scalar to distinguish it from a vector.)

Definition 2 Let λ be a nonzero scalar and **a** a nonzero vector. Then the **product** (or scalar multiple) of λ and **a** is a vector, denoted by $\lambda\,\mathbf{a}$. Its length is $|\lambda\,\mathbf{a}|=|\lambda|\,|\mathbf{a}|$, its direction is the same as that of **a** if $\lambda>0$ or is opposite to that of **a** if $\lambda<0$. If $\lambda=0$ or $\mathbf{a}=\mathbf{0}$, we define $\lambda\,\mathbf{a}=\mathbf{0}$. From Definition 2 we have $1\mathbf{a}=\mathbf{a}$, $(-1)\mathbf{a}=-\mathbf{a}$ and $\mathbf{a}=|\mathbf{a}|\mathbf{e}_a$, where $\mathbf{e}_a$ is the unit vector in the direction of **a**. We set $\frac{\mathbf{a}}{\lambda}=\frac{1}{\lambda}\mathbf{a}$ when $\lambda\neq0$, so that we have $\frac{\mathbf{a}}{|\mathbf{a}|}=\mathbf{e}_a$, provided $\mathbf{a}\neq\mathbf{0}$. It tells us that the result of a nonzero vector divided by its length is a vector in the same direction of the original vector.

Products of scalars and vectors satisfy the following laws.

① Associative law $\lambda(\mu\mathbf{a})=\mu(\lambda\mathbf{a})=(\lambda\mu)\mathbf{a}$;

According to Definition 2, vectors $\lambda(\mu\mathbf{a})$, $\mu(\lambda\mathbf{a})$ and $(\lambda\mu)\mathbf{a}$ have the same direction, and

$$|\lambda(\mu\mathbf{a})|=|\mu(\lambda\mathbf{a})|=|(\lambda\mu)\mathbf{a}|=|\lambda\mu|\,|\mathbf{a}|$$

Therefore, $\lambda(\mu\mathbf{a})=\mu(\lambda\mathbf{a})=(\lambda\mu)\mathbf{a}$.

② Distributive law $(\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a}$; $\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}$.

We prove only the distributive law $\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}$, leaving the other to readers. If $\lambda=0$, the equality holds obviously. Let $\lambda>0$ and draw $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\lambda\mathbf{a}$, $\overrightarrow{AD}=\mathbf{b}$, $\overrightarrow{BE}=\lambda\mathbf{b}$ (Figure 8-7(c)). Then the three points O,A,B are collinear, and $\overrightarrow{AD}//\overrightarrow{BE}$. Therefore, $\frac{|\overrightarrow{OE}|}{|\overrightarrow{OD}|}=\frac{|\overrightarrow{OB}|}{|\overrightarrow{OA}|}=\lambda$ and the points O,D,E are also collinear, $\overrightarrow{OE}=\lambda\,\overrightarrow{OD}$. According to the triangle law, we have $\overrightarrow{OE}=\lambda\mathbf{a}+\lambda\mathbf{b}$, $\overrightarrow{OD}=\mathbf{a}+\mathbf{b}$, and hence $\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}$. If $\lambda<0$, the proof is similar.

Vector addition and scalar multiplication are called by a joint name vector **linear operations.**

From the above discussion we know that the length of a vector has the following basic properties.

① Nonnegativity $|\mathbf{a}|\geqslant0$, and $|\mathbf{a}|=0\Leftrightarrow\mathbf{a}=0$.

② Absolute homogeneity $|\lambda\,\mathbf{a}|=|\lambda|\,|\mathbf{a}|$.

③ Triangle inequality $|\mathbf{a}+\mathbf{b}|\leqslant|\mathbf{a}|+|\mathbf{b}|$, where the sign of equality holds $\Leftrightarrow$ **a** and **b** have the same direction.

The geometric meaning of the triangle inequality is that the sum of the lengths of two adjoining sides of a triangle is greater than or equal to the length of the third side of the triangle.

Example 1 The accompanying Figure 8-8 shows parallelogram $ABCD$ and the midpoint M of diagonal BD. Let $\overrightarrow{AB}=\mathbf{a}$, $\overrightarrow{AD}=\mathbf{b}$. Express $\overrightarrow{MA}$, $\overrightarrow{MB}$, $\overrightarrow{MC}$ and $\overrightarrow{MD}$ in terms of **a** and **b**.

Solution. Because the two diagonals of a parallelogram bisect each other, so $\mathbf{a}+\mathbf{b}=\overrightarrow{AC}=2\,\overrightarrow{AM}$, that is $-(\mathbf{a}+\mathbf{b})=2\,\overrightarrow{MA}$. Thus, $\overrightarrow{MA}=-\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

Since $\overrightarrow{MC}=-\overrightarrow{MA}$, we have $\overrightarrow{MC}=\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

Since $-\mathbf{a}+\mathbf{b}=\overrightarrow{BD}=2\overrightarrow{MD}$, we have $\overrightarrow{MD}=\frac{1}{2}(\mathbf{b}-\mathbf{a})$.

Since $\overrightarrow{MB}=-\overrightarrow{MD}$, we have $\overrightarrow{MB}=\frac{1}{2}(\mathbf{a}-\mathbf{b})$.

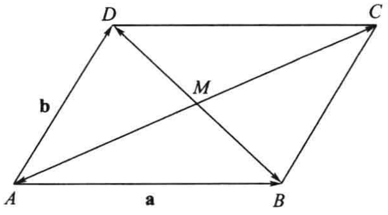

Figure 8-8

Theorem 1 Suppose the vector $\mathbf{a}\neq\mathbf{0}$, then the sufficient and necessary condition of that $\mathbf{b}$ is parallel to $\mathbf{a}$ is that there exists only one real number λ such that $\mathbf{b}=\lambda\mathbf{a}$.

Proof. Sufficiency of the condition is obvious, so we prove the necessity of the condition in the following part.

Suppose $\mathbf{b}//\mathbf{a}$. Choose $|\lambda|=\frac{|\mathbf{b}|}{|\mathbf{a}|}$. When $\mathbf{b}$ and $\mathbf{a}$ have the same direction, then $\lambda>0$. When $\mathbf{b}$ and $\mathbf{a}$ have the inverse direction, then $\lambda<0$, i. e. $\mathbf{b}=\lambda\,\mathbf{a}$, here, $\mathbf{b}$ and $\lambda\,\mathbf{a}$ have the same direction, and $|\lambda\mathbf{a}|=|\lambda|\,|\mathbf{a}|=\frac{|\mathbf{b}|}{|\mathbf{a}|}|\mathbf{a}|=|\mathbf{b}|$.

Now we prove that λ is unique. Suppose $\mathbf{b}=\lambda\mathbf{a}$ and $\mathbf{b}=\mu\mathbf{a}$, make the difference of the two, we have $(\lambda-\mu)\mathbf{a}=\mathbf{0}$, i. e. $|\lambda-\mu|\,|\mathbf{a}|=0$.

Since $|\mathbf{a}|\neq 0$, $|\lambda-\mu|=0$, i. e. $\lambda=\mu$. This is the end of the proof of Theorem 1.

Figure 8-9

Theorem 1 is the theoretical basis of establishing number line. As we have known, a given point, a given direction and unit length determine a number line. Since a unit vector determines not only a direction but also a unit length, hence a given point and a unit vector determine a number line. Suppose that a point O and a unit vector $\mathbf{i}$ determine a number line Ox (Figure 8-9). For any point P on the number line, there is a vector $\overrightarrow{OP}$ corresponding to it. Since $\overrightarrow{OP}//\mathbf{i}$, by Theorem 1, there is a unique real number x such that $\overrightarrow{OP}=x\mathbf{i}$ (the real number x is called the value of directed line segment $\overrightarrow{OP}$ on the number line), $\overrightarrow{OP}$ and x are one-to-one. Thus,

$$\text{Point } P \leftrightarrow \text{Vector } \overrightarrow{OP}=x\mathbf{i} \leftrightarrow \text{Real number } x$$

It shows us that the point P and the real number x are one-to-one. We can define the real number x as the coordinate of P, and $\overrightarrow{OP}=x\mathbf{i}$ is the sufficient and necessary condition of that the coordinate of point P is x.

8.1.3 Three-dimensional rectangular coordinate system

We choose a fixed point O in space and three number lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis respectively. The three axes have the same origin O and the same unit of length. Usually we think of the x-and y-axes as being horizontal and the z-axis as being vertical, and we draw the orientation of the axes as in Figure 8-10. The direction of the z-axis is determined by the **right-hand rule** as illustrated in Figure 8-11: If you curl the fingers of your right hand around the z-axis in the direction of a $\frac{\pi}{2}$ counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

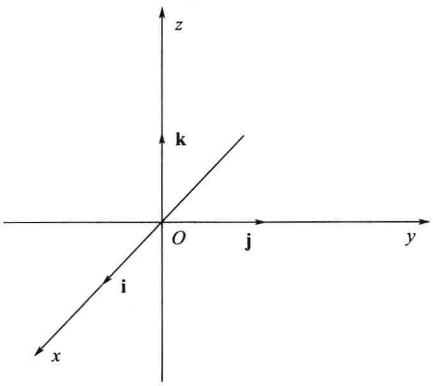

Figure 8-10

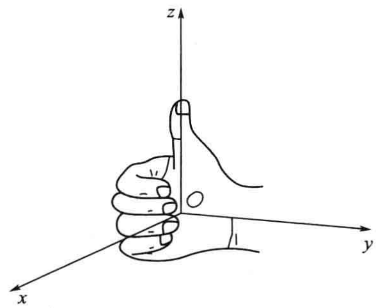

Figure 8-11

These three coordinate axes make a **three-dimensional rectangular coordinate system** denoted by $Oxyz$ or $[O;\mathbf{i},\mathbf{j},\mathbf{k}]$; the point O is called the **coordinate origin** or **origin** (Figure 8-10). The three unit vectors on the x-axis, y-axis and z-axis with the same directions as the corresponding axes are called **basic unit vectors** and denoted by $\mathbf{i},\mathbf{j},\mathbf{k}$ respectively.

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 8-12 (a). The xOy plane is the plane that contains the x-axis and y-axis; the yOz plane contains the y-axis and z-axis; the zOx plane contains the z-axis and x-axis. These three coordinate planes divide space into eight parts, called **octants**, where the octants Ⅰ, Ⅱ, Ⅲ, Ⅳ lie over the quadrants 1, 2, 3, 4 of the xOy plane respectively, and the octants Ⅴ, Ⅵ, Ⅶ, Ⅷ lie below the quadrants 1, 2, 3, 4 of the xOy plane respectively (Figure 8-12(b)).

Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following (Figure 8-12(c)). Look at any bottom corner of a room and call the corner the origin. The wall on your left is in the zOx plane, the wall on your right is in the yOz plane, and the floor is in the xOy plane. The x-axis runs along the intersection of the floor and the left wall. The y-axis runs along the intersection of the floor and the right wall. The z-axis runs up from the floor toward the ceiling along the intersection of the two walls. You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point O.

Now if M is any point in space, we draw three planes through M perpendicular to the x-axis, y-axis, and z-axis respectively. The points of intersection with the axes are P, Q, R respectively (Figure 8-13). The points P, Q and R are called the **projections** of M on the x-axis, y-axis, and z-axis respectively. Suppose that these three projections have coordinates x, y and z on the x-axis, y-axis, and z-axis respectively. Then the ordered triple (x, y, z) of real numbers is determined uniquely by the point M. Conversely, for a given ordered triple (x, y, z) of real numbers, we take a point P whose coordinate is x on the x-axis, take a point Q whose coordinate is y on the y-axis, and take a point R whose coordinate is z on the z-axis. We draw three planes through P, Q and R which are perpendicular to the x-axis, y-axis, and z-axis respectively. The point of intersection M of these three planes is a unique point in space determined by the ordered triple$(x, y,$

z). We have now given a one-to-one correspondence between point M in space and ordered triple (x, y, z) of real numbers. This ordered triple (x, y, z) is called the set of **coordinates** of M, denoted by $M(x, y, z)$, and x, y, z are called the **abscissa**, **ordinate** and **vertical coordinate** of M respectively.

8.1.3.1 The radius vector and its components

Any nonzero vector $\overrightarrow{OM}$ with initial point at the origin O is called the **radius vector** of the point M, or radius vector $\overrightarrow{OM}$. We know that a vector can be moved parallel to itself, but a radius vector is a special vector whose initial point is fixed at the origin. It is easy to see from Figure 8-13 that the projection vectors of the radius vector $\overrightarrow{OM}$ onto the basic unit vectors **i**, **j**, **k** (or onto the x-axis, y-axis and z-axis) are $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$ respectively, and we have $\overrightarrow{OM}=\overrightarrow{OP}+\overrightarrow{PN}+\overrightarrow{NM}=\overrightarrow{OP}+\overrightarrow{OQ}+\overrightarrow{OR}$, that is,

$$\overrightarrow{OM}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \qquad (8\text{-}1)$$

where (x, y, z) is just the set of coordinates of the terminal M of the radius vector $\overrightarrow{OM}$.

The Formula (8-1) is called the **decomposition** of the radius vector with respect to the basic unit vectors **i**, **j** and **k**. Obviously, when the point M is given, the above decomposition of $\overrightarrow{OM}$ is unique; conversely, the radius vector $\overrightarrow{OM}$ is determined uniquely by its decomposition. The ordered triple (x, y, z) is called coordinates of the radius vector $\overrightarrow{OM}$, or **components of the radius vector** $\overrightarrow{OM}$. It can be seen that the components of a radius vector $\overrightarrow{OM}$ are just the coordinates of the terminal point M of the radius vector and vice versa.

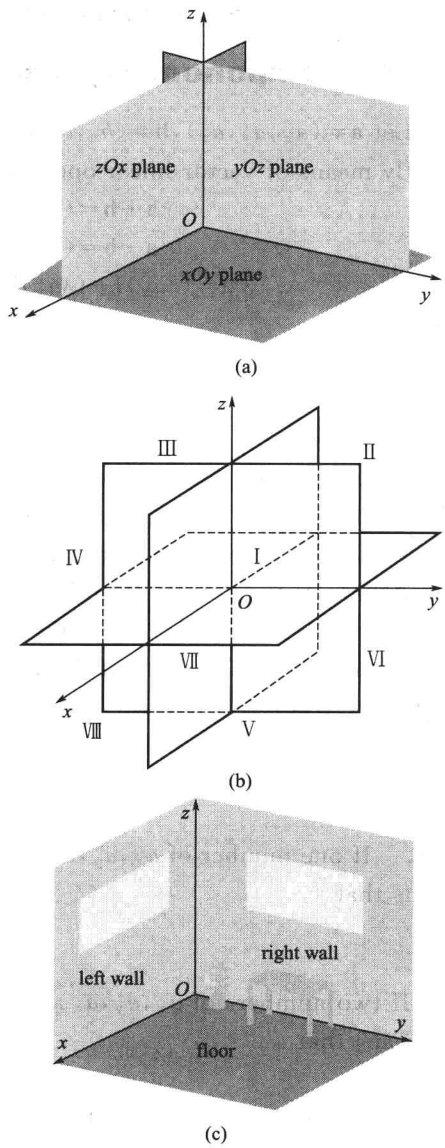

Figure 8-12

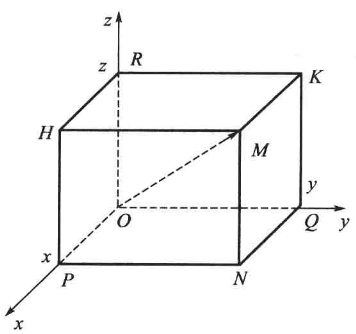

Figure 8-13

8.1.3.2 Components of a vector

Let **a** be any nonzero vector in a three-dimensional rectangular coordinate system. We move **a** parallel to itself such that its initial point is at the origin. Its terminal point is denoted by $M(x, y, z)$; thus $\mathbf{a}=\overrightarrow{OM}$. The components (x, y, z) of the radius vector $\overrightarrow{OM}$ are defined as the components of the vector **a**, denoted by $\mathbf{a}=(x, y, z)$. x, y and z are called the **first**, **second** and **third component** (or **coordinate**) of the vector **a**. Then

$$\mathbf{a}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

is called the **component decomposition** of the vector **a**.

8.1.4 Component representation of vector linear operations

Let $\mathbf{a}=(a_x,a_y,a_z)$, $\mathbf{b}=(b_x,b_y,b_z)$, that is $\mathbf{a}=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}$, $\mathbf{b}=b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k}$.

By means of vector linear operations we get

$$\mathbf{a}+\mathbf{b}=(a_x+b_x)\mathbf{i}+(a_y+b_y)\mathbf{j}+(a_z+b_z)\mathbf{k}$$
$$\mathbf{a}-\mathbf{b}=(a_x-b_x)\mathbf{i}+(a_y-b_y)\mathbf{j}+(a_z-b_z)\mathbf{k}$$
$$\lambda\,\mathbf{a}=(\lambda a_x)\mathbf{i}+(\lambda a_y)\mathbf{j}+(\lambda a_z)\mathbf{k}\quad (\lambda \text{ is a real number here})$$

That is,

$$\begin{aligned}\mathbf{a}+\mathbf{b}&=(a_x+b_x,a_y+b_y,a_z+b_z)\\ \mathbf{a}-\mathbf{b}&=(a_x-b_x,a_y-b_y,a_z-b_z)\\ \lambda\mathbf{a}&=(\lambda a_x,\lambda a_y,\lambda a_z)\end{aligned}\tag{8-2}$$

In other words, the addition(difference)operation for two vectors is the same as the addition (difference)of their corresponding components; the operation of multiplication of a scalar and a vector is the same as the operation of multiplication of the scalar and the corresponding components of the vector. From Formula(8-2), we have $\mathbf{a}=\mathbf{b}\Leftrightarrow a_x=b_x,a_y=b_y,a_z=b_z$. That is, two vectors are equal if and only if their corresponding components are equal.

From Theorem 1, it is not difficult to obtain that, **a** and **b** are collinear(or parallel) if and only if there is a real number λ, such that $\mathbf{b}=\lambda\mathbf{a}$. That is, $(b_x,b_y,b_z)=\lambda(a_x,a_y,a_z)$. Then

$$\frac{b_x}{a_x}=\frac{b_y}{a_y}=\frac{b_z}{a_z}\tag{8-3}$$

Note. If one number of a_x,a_y,a_z is zero, such as $a_x=0,a_y,a_z\neq0$, then the Formula (8-3) means that

$$b_x=0,\quad \frac{b_y}{a_y}=\frac{b_z}{a_z}$$

If two numbers of a_x,a_y,a_z are zeros, such as $a_x=a_y=0,a_z\neq0$, then the Formula (8-3) means that

$$b_x=0,\quad b_y=0$$

The meanings are similar for the other cases.

Example 2 Solve the following linear system of equations whose unknowns are vectors:

$$\begin{cases}5\mathbf{x}-3\mathbf{y}=\mathbf{a},\\ 3\mathbf{x}-2\mathbf{y}=\mathbf{b}\end{cases}$$

where $\mathbf{a}=(2,1,2)$, $\mathbf{b}=(-1,1,-2)$.

Solution. As solving linear system of equations whose unknowns are real numbers, we find

$$\mathbf{x}=2\mathbf{a}-3\mathbf{b},\quad \mathbf{y}=3\mathbf{a}-5\mathbf{b}$$

Substituting the components representation of **a** and **b** into above formulas, then we get

$$\mathbf{x}=2(2,1,2)-3(-1,1,-2)=(7,-1,10)$$
$$\mathbf{y}=3(2,1,2)-5(-1,1,-2)=(11,-2,16)$$

Example 3 Suppose that $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ are two given points. Find a point M on the line segment AB such that

$$\overrightarrow{AM}=\lambda\overrightarrow{MB}\quad(\lambda\neq-1)$$

Solution. As shown in Figure 8-14, since