Third Edition

Elements of Advanced Mathematics

Steven G. Krantz

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A CHAPMAN & HALL BOOK

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To the memory of R. P. Boas, 1912–1992.
For his commitment to mathematics, and for the clarity of his vision.

Preface to the Third Edition

On the whole, we have retained the content and character of the first two editions. But we have added material on point-set topology (Chapter 8), on theoretical computer science (Chapter 9), on the **P/NP** problem (Chapter 10), and on zero-knowledge proofs and RSA encryption (Chapter 12). The topology chapter, of course, builds on the existing material on real analysis. The computer science chapters show connections of basic set theory and logic with current hot topics in the technology sector. The material on cryptography is exciting, timely, and fun. These new chapters help to make the book more current and significant. It should of course be understood that these four chapters may be considered to be optional. Skipping them will in no way detract from reading the rest of the book.

Some readers consider Chapter 5 on axiomatics and rigorous logic to be optional. To be sure, it is a more demanding chapter than some of the others. But it contains important material, some of which is at least alluded to later in the book. Readers who do not want to spend much time on Chapter 5 might wish to at least have a look at it.

The main message here is that Chapters 5, 8, 9, 10, and 12 provide an open-ended venue for students to explore and to learn. My experience with teaching this course is that the aggregate material causes many of the students to get really turned on to mathematics. They need to have a means for further exploration and reading. These chapters give them that opportunity, and exercises to back up the reading.

The new Chapter 12 is dessert. It presents the very new ideas of zero-knowledge proofs and RSA encryption. A lovely application of elementary groups theory (which is introduced in Chapter 11) and logic, these ideas are at the cutting edge of modern cryptography and security analysis. If students want to see what mathematics is good for, this is grist for their mill.

We have also beefed up the exercise sets in all the chapters. We have expanded the treatment of proofs, and added some new proof techniques. Of course errors and omissions in the existing chapters have been handled, and the text as a whole has been polished and improved.

We are happy for the positive reception that this book has received, and look forward to further interactions with the readers.

SGK St. Louis, Missouri

Preface to the Second Edition

Prologue

The audience for a "transitions course" for mathematics students continues to grow. The gap between the rote, calculational learning mode of calculus and ordinary differential equations and the more theoretical learning mode of analysis and abstract algebra is ever more distinct. A pathway is needed to help students to understand rigor, axiomatics, set theory, and proofs. Especially because the modern high school curriculum is ever more lacking in these traditional mathematical artifacts, the need for a transitions book is increasingly pronounced.

The present book has been well received and widely used. It is a pleasure to have this opportunity to update the book, to add new material, and to correct some errors. We have augmented the references and the exercises, and we have expanded the scope of some sections.

New Topics and Revisions

New topics include

- Further explanation of propositional logic, predicate logic, first-order logic, and related ideas. Especially in view of the prominence of theoretical computer science and its symbiotic relationship with logic, we feel that it is important to develop these ideas fully.

- Construction and discussion of the nonstandard real numbers.

- A new chapter that explores deeper properties of the real numbers. This includes topological issues and the construction of the Cantor set.

- A more exhaustive treatment of proof techniques. We have added more on induction, on counting arguments, and on enumeration and dissection. There are more geometric proofs.

- A more thorough treatment of the Axiom of Choice and its equivalents. This includes a discussion of the Banach–Tarski paradox.

- An explicit discussion of Zorn's lemma, the Hausdorff maximality principle, and other equivalents of the Axiom of Choice.
- A treatment of partial orderings, total orderings, and well orderings. These fit rather naturally into the context of our treatment of relations.
- A fuller discussion of independence and consistency. Again, students with an interest in computer science may especially appreciate this material.
- Additional material on Russell's paradox and related ideas.
- Additional material on group theory. Group theory is an ideal venue in which students may experience the axiomatic method for the first time, and we endeavor to make the most of it.
- A more streamlined treatment of non-Euclidean geometry. Our discussion of this topic differs from other books in the marketplace. But we feel that this material can get rather technical rather quickly, and we have endeavored to make this section of the book as slick as possible.

The book has a large number of figures, diagrams, and tables. We feel that such devices tend to bring the material to life for students, and serve well to organize logical ideas. There are an especially large number of exercises. The typical instructor may assign only a dozen from any given chapter, but the ambitious student will want to test his or her mettle against many of the others. Many of the solutions to the exercises are in the back of the book. The Solutions Manual provides additional solutions.

We hope that this new edition will be useful and enjoyable for student and instructor alike. The book has become useful not only for mathematics students but also for those studying theoretical computer science and other sciences. It has served to attract new mathematics majors, and we hope that it will continue to do so.

As always, the author bears full responsibility for any remaining errors or malapropisms. Comments and criticisms are welcome.

— Steven G. Krantz
St. Louis, Missouri

Preface to the First Edition

Overview

The character of lower division mathematics courses in universities in the United States is, and should be, different from that of upper division mathematics courses. Oversimplifying a bit, we might say that lower division courses concentrate on technique while upper division courses treat theory.

In order to achieve any depth, an upper division mathematics course must use a precise language and methodology. The standard mathematical language includes logic, set theory, the use of functions, equivalence relations, rigorous proofs, axiomatic structures, and so forth. We frequently find ourselves, when teaching an upper division mathematics course, giving a whirlwind treatment of these basic ideas during the first week or two of the class; we also find ourselves playing catch-up during the remainder of the term.

Such a practice results in needless repetition of these common tools. It seems logical, and practical, to give the student a considered exposure to these ideas once and for all, before upper division work is commenced. That is the purpose of the present book.

Audience

Let me stress that this is not, in the strict sense, a book of logic; nor is it a book of set theory. Logicians may disapprove of my dismissal of certain subtleties. For instance, I shall not compare the merits of various versions of set theory, nor shall I discuss attempts (such as Martin's axiom) to work around the independence of the continuum hypothesis. Rather, my purpose is to give the student, typically a second semester sophomore or first semester junior, a quick introduction to *one version* of the foundations of mathematics. In short, this is not a book for mathematicians; it is a book for students.

The student who has spent a semester studying this book should, in principle, be properly prepared for a course in real analysis or elementary Riemannian geometry or abstract algebra. Of course there is no substitute for mathematical sophistication and hard work. This book merely provides the student with the tools of the trade.

Prerequisites for this book are minimal. Formally, the only prerequisite is an ability to read English. But, truth be told, a certain amount of exposure to mathematics

and mathematical methodology is recommended. Every chapter contains a significant number of exercises. These are not merely window dressing; they are essential for mastery of the material. The student should do as many of these as possible.

Using This Book in Class

And now a few remarks about the layout of the book. A typical course will cover Chapters 1 through 4 rather thoroughly. These cover the basics of logic, proofs, set theory, relations, functions, and cardinality. Chapter 5 is rather sophisticated (for a book at this level), and the instructor will want to exercise discretion when treating this chapter. The primary message of the chapter is the importance of axiomatics. It is also possible to skip Chapter 5 altogether. Chapter 6 is one of the main points of the book: to construct the number systems that we use in mathematics. This material is a good venue for the student to sharpen the skills and ideas developed in earlier chapters. Some instructors may wish to skip the complex numbers, the quaternions, and the Cayley numbers, but the basic number systems should certainly be treated in detail. Chapter 7, the closing chapter of the book, is dessert. Here we give a quick treatment of two important axiomatic systems in mathematics: groups, and Euclidean and non-Euclidean geometries. This instructor has found students quite receptive to, and interested in, both these topics. But sufficient time must be available to give them proper treatment.

Acknowledgments

I am grateful to Harold P. Boas for reading an early draft of the manuscript of the book and contributing numerous remarks and suggestions. Joseph A. Cima, C. David Minda, and Harold R. Parks served as reviewers of the manuscript for CRC Press. I am happy to thank them for their ideas and contributions.

I developed this material while teaching a course on the subject of mathematical foundations at Washington University in St. Louis. I thank the university and the mathematics department for giving me this opportunity.

— Steven G. Krantz
St. Louis, Missouri

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Chapter 1

Basic Logic

1.1 Principles of Logic

Strictly speaking, our approach to logic is "intuitive" or "naïve". Whereas in ordinary conversation these emotion-charged words may be used to downgrade the value of that which is being described, our use of these words is more technical. What is meant is that we shall prescribe in this chapter certain rules of logic which are to be followed in the rest of the book. They will be presented to you in such a way that their validity should be intuitively appealing and self-evident. We cannot *prove* these rules. The rules of logic are the point where our learning begins. A more advanced course in logic will explore other logical methods. The ones that we present here are universally accepted in mathematics and in most of science.

We shall begin with sentential logic and elementary connectives. This material is called the *propositional calculus* (to distinguish it from the predicate calculus, which will be treated later). In other words, we shall be discussing *propositions*—which are built up from atomic statements and connectives. The elementary connectives include "and," "or," "not," "if-then," and "if and only if." Each of these will have a precise meaning and will have exact relationships with the other connectives. In Section 1.8 we shall discuss the completeness of this system of elementary sentential logic, although we shall not present the *proof* of completeness (see [STO, p. 147 ff.] for a discussion of the work of Frege, Whitehead and Russell, Bernays, and Gödel in this regard).

An *atomic statement* (or *elementary statement*) is a sentence with a subject and a verb (and sometimes an object) but no connectives (and,

or, not, if-then, if-and-only-if). For example,

John is good.

Mary has bread.

Ethel reads books.

are all atomic statements. We build up sentences, or propositions, from atomic statements using connectives.

Next we shall consider the quantifiers "for all" and "there exists" and their relationships with the connectives from the last paragraph. The quantifiers will give rise to the so-called *predicate calculus*. Connectives and quantifiers will prove to be the building blocks of all future statements in this book, indeed in all of mathematics.

1.2 Truth

In everyday conversation, people sometimes argue about whether a statement is true or not. In mathematics there is nothing to argue about. In practice a sensible statement in mathematics is either true or false, and there is no room for opinion about this attribute. How do we determine which statements are true and which are false?

The modern methodology in mathematics works as follows:

- We *define* certain terms.
- We *assume* that these terms have certain properties or truth attributes (these assumptions are called axioms).
- We specify certain rules of logic.

Any statement that can be derived from the axioms, using the rules of logic, is understood to be true (we call such a derivation a *proof*). It is not necessarily the case that every true statement can be derived in this fashion. However, in practice this is our method for verifying that a statement is true. See Section 1.8 for a more detailed discussion of truth versus provability.

On the other hand, a statement is false if it is inconsistent with the axioms and the rules of logic. That is to say, a statement is false if the assumption that it is true leads to a contradiction. Alternatively, a statement **P** is false if the negation of **P** can be established or proved.

While it is possible for a statement to be false without our being able to derive a contradiction in this fashion, in practice we establish falsity by the method of contradiction or by giving a counterexample (which is another aspect of the method of contradiction). Again, see Section 1.8 for more on falsity versus inconsistency.

The point of view being described here is special to mathematics. While it is indeed true that mathematics is used to model the world around us—in physics, engineering, and in other sciences—the subject of mathematics itself is a man-made system. Its internal coherence is guaranteed by the axiomatic method that we have just described.

It is reasonable to ask whether mathematical truth is a construct of the human mind or an immutable part of nature. For instance, is the assertion that "the area of a circle is π times the radius squared" actually a fact of nature just like Newton's inverse square law of gravitation? Our point of view is that mathematical truth is relative. The formula for the area of a circle is a logical consequence of the axioms of mathematics, nothing more. The fact that the formula seems to describe what is going on in nature is convenient, and is part of what makes mathematics useful. But that aspect is something over which we as mathematicians have no control. Our concern is with the internal coherence of our logical system.

It can be asserted that a proof (a concept to be discussed and developed later in the book) is a psychological device for convincing the reader that an assertion is true. However, our view in this book is more rigid: a proof is a sequence of applications of the rules of logic to derive the assertion from the axioms. There is no room for opinion here. The axioms are plain. The rules are rigid. A proof is like a sequence of moves in a game of chess. If the rules are followed, then the proof is correct; otherwise not.

1.3 "And" and "Or"

Let **A** and **B** be atomic statements such as "Chelsea is smart" or "The earth is flat." The statement

"A and B"

means that both **A** is true *and* **B** is true. For instance,

Arvid is old and Arvid is fat.

means both that Arvid is old *and* Arvid is fat. If we meet Arvid and he turns out to be young and fat, then the statement is false. If he is old and thin then the statement is false. Finally, if Arvid is *both* young and thin then the statement is false. The statement is *true* precisely when both properties—oldness and fatness—hold. We may summarize these assertions with a *truth table*. We let

A = Arvid is old.

and

B = Arvid is fat.

The expression

$$\mathbf{A} \wedge \mathbf{B}$$

will denote the phrase "**A and B**". We call this statement the *conjunction* of **A** and **B**. The letters "T" and "F" denote "True" and "False," respectively. Then we have

A	**B**	**A∧B**
T	T	T
T	F	F
F	T	F
F	F	F

Notice that we have listed all possible truth values of **A** and **B** and the corresponding values of the *conjunction* **A∧B**.

In a restaurant, the menu often contains phrases such as

soup or salad

This means that we may select soup *or* select salad, but we may not select both. This use of "or" is called the *exclusive* "or"; it is not the meaning of "or" that we use in mathematics and logic. In mathematics we instead say that "**A or B**" is true provided that **A** is true or **B** is true or *both* are true. This is the *inclusive* "or." If we let **A∨B** denote "**A or B**" then the truth table is

A	**B**	**A∨B**
T	T	T
T	F	T
F	T	T
F	F	F