A COURSE
OF
PURE MATHEMATICS

BY

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PREFACE TO THE TENTH EDITION

THE changes in the present edition are as follows:

1. An index has been added. Hardy had begun a revision of an index compiled by Professor S. Mitchell; this has been completed, as far as possible on Hardy's lines, by Dr T. M. Flett.

2. The original proof of the Heine-Borel Theorem (pp. 197–199) has been replaced by two alternative proofs due to Professor A. S. Besicovitch.

3. The 'Implicit Function Theorem' (p. 203) has now a revised statement and proof due to Professor A. S. Besicovitch.

4. Example 24, p. 394 has been added to.

August, 1950 J. E. LITTLEWOOD

PREFACE TO THE SEVENTH EDITION

THE changes in this edition are more important than in any since the second. The book has been reset, and this has given me the opportunity of altering it freely.

I have cancelled what was Appendix II (on the 'O, o, $\sim$' notation), and incorporated its contents in the appropriate places in the text. I have rewritten the parts of Chs. VI and VII which deal with the elementary properties of differential coefficients. Here I have found de la Vallée-Poussin's *Cours d'analyse* the best guide, and I am sure that this part of the book is much improved. These important changes have naturally involved many minor emendations.

I have inserted a large number of new examples from the papers for the Mathematical Tripos during the last twenty years, which should be useful to Cambridge students. These were collected for me by Mr E. R. Love, who has also read all the proofs and corrected many errors.

The general plan of the book is unchanged. I have often felt tempted, re-reading it in detail for the first time for twenty years, to make much more drastic changes both in substance and in style. It was written when analysis was neglected in Cambridge, and with an emphasis and enthusiasm which seem rather ridiculous now. If I were to rewrite it now I should not write (to use Prof. Littlewood's simile) like 'a missionary talking to cannibals', but with decent terseness and restraint; and, writing more shortly, I should be able to include a great deal more. The book would then be much more like a *Traité d'analyse* of the standard pattern.

It is perhaps fortunate that I have no time for such an undertaking, since I should probably end by writing a much better but much less individual book, and one less useful as an introduction to the books on analysis of which, even in England, there is now no lack.

November, 1937 G. H. H.

EXTRACT FROM THE PREFACE TO THE FIRST EDITION

THIS book has been designed primarily for the use of first year students at the Universities whose abilities reach or approach something like what is usually described as 'scholarship standard'. I hope that it may be useful to other classes of readers, but it is this class whose wants I have considered first. It is in any case a book for mathematicians: I have nowhere made any attempt to meet the needs of students of engineering or indeed any class of students whose interests are not primarily mathematical.

I regard the book as being really elementary. There are plenty of hard examples (mainly at the ends of the chapters): to these I have added, wherever space permitted, an outline of the solution. But I have done my best to avoid the inclusion of anything that involves really difficult ideas.

September, 1908 G. H. H.

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CHAPTER I

REAL VARIABLES

1. Rational numbers. A fraction $r = p/q$, where p and q are positive or negative integers, is called a *rational number*. We can suppose (i) that p and q have no common factor, since if they have a common factor we can divide each of them by it, and (ii) that q is positive, since

$$p/(-q) = (-p)/q, \quad (-p)/(-q) = p/q.$$

To the rational numbers thus defined we may add the 'rational number 0' obtained by taking $p = 0$.

We assume that the reader is familiar with the ordinary arithmetical rules for the manipulation of rational numbers. The examples which follow demand no knowledge beyond this.

Examples I. 1. If r and s are rational numbers, then $r+s$, $r-s$, rs, and r/s are rational numbers, unless in the last case $s = 0$ (when r/s is of course meaningless).

2. If λ, m, and n are positive rational numbers, and $m > n$, then $\lambda(m^2-n^2)$, $2\lambda mn$, and $\lambda(m^2+n^2)$ are positive rational numbers. Hence show how to determine any number of right-angled triangles the lengths of all of whose sides are rational.

3. Any terminated decimal represents a rational number whose denominator contains no factors other than 2 or 5. Conversely, any such rational number can be expressed, and in one way only, as a terminated decimal.

[The general theory of decimals will be considered in Ch. IV.]

4. The positive rational numbers may be arranged in the form of a simple series as follows:

$$\tfrac{1}{1}, \tfrac{2}{1}, \tfrac{1}{2}, \tfrac{3}{1}, \tfrac{2}{2}, \tfrac{1}{3}, \tfrac{4}{1}, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}, \ldots.$$

Show that p/q is the $[\frac{1}{2}(p+q-1)(p+q-2)+q]$th term of the series.

[In this series every rational number is repeated indefinitely. Thus 1 occurs as $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots$. We can of course avoid this by omitting every number which has already occurred in a simpler form, but then the problem of determining the precise position of p/q becomes more complicated.]

2. The representation of rational numbers by points on a line. It is convenient, in many branches of mathematical analysis, to make a good deal of use of geometrical illustrations.

The use of geometrical illustrations in this way does not, of course, imply that analysis has any sort of dependence upon geometry: they are illustrations and nothing more, and are employed merely for the sake of clearness of exposition. This being so, it is not necessary that we should attempt any logical analysis of the ordinary notions of elementary geometry; we may be content to suppose, however far it may be from the truth, that we know what they mean.

Assuming, then, that we know what is meant by a *straight line*, a *segment* of a line, and the *length* of a segment, let us take a straight line Λ, produced indefinitely in both directions, and a segment A_0A_1 of any length. We call A_0 the *origin*, or *the point* 0, and A_1 *the point* 1, and we regard these points as representing the numbers 0 and 1.

In order to obtain a point which shall represent a positive rational number $r = p/q$, we choose the point A_r such that

$$A_0A_r/A_0A_1 = r,$$

A_0A_r being a stretch of the line extending in the same direction along the line as A_0A_1, a direction which we shall suppose to be from left to right when, as in Fig. 1, the line is drawn horizontally across the paper. In order to obtain a point to represent a

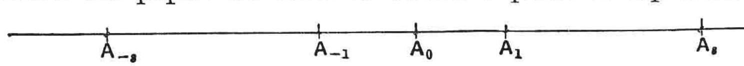

Fig. 1

negative rational number $r = -s$, it is natural to regard length as a magnitude capable of sign, positive if the length is measured in one direction (that of A_0A_1), and negative if measured in the other, so that $AB = -BA$; and to take as the point representing r the point A_{-s} such that

$$A_0A_{-s} = -A_{-s}A_0 = -A_0A_s.$$

We thus obtain a point A_r on the line corresponding to every rational value of r, positive or negative, and such that

$$A_0A_r = r.A_0A_1;$$

and if, as is natural, we take A_0A_1 as our unit of length, and write $A_0A_1 = 1$, then we have

$$A_0A_r = r.$$

We shall call the points A_r the *rational points* of the line.

3. Irrational numbers. If the reader will mark off on the line all the points corresponding to the rational numbers whose denominators are 1, 2, 3, ... in succession, he will readily convince himself that he can cover the line with rational points as closely as he likes. We can state this more precisely as follows: *if we take any segment BC on Λ, we can find as many rational points as we please on BC.*

Suppose, for example, that BC falls within the segment A_1A_2. It is evident that if we choose a positive integer k so that

$$k.BC > 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)^*,$$

and divide A_1A_2 into k equal parts, then at least one of the points of division (say P) must fall inside BC, without coinciding with either B or C. For if this were not so, BC would be entirely included in one of the k parts into which A_1A_2 has been divided, which contradicts the supposition (1). But P obviously corresponds to a rational number whose denominator is k. Thus at least one rational point P lies between B and C. But then we can find another such point Q between B and P, another between B and Q, and so on indefinitely; i.e., as we asserted above, we can find as many as we please. We may express this by saying that BC includes *infinitely many* rational points.

The meaning of such phrases as '*infinitely many*' or '*an infinity of*', in such sentences as 'BC includes infinitely many rational points' or 'there are an infinity of rational points on BC' or 'there are an infinity of positive integers', will be considered more closely in Ch. IV. The assertion 'there are an infinity of positive integers' means 'given any positive integer n,

* The assumption that this is possible is equivalent to the assumption of what is known as the axiom of Archimedes.

however large, we can find more than n positive integers'. This is plainly true whatever n may be, e.g. for $n = 100{,}000$ or $100{,}000{,}000$. The assertion means exactly the same as 'we can find *as many positive integers as we please*'.

The reader will easily convince himself of the truth of the following assertion, which is substantially equivalent to what was proved in the second paragraph of this section: given any rational number r, and any positive integer n, we can find another rational number lying on either side of r and differing from r by less than $1/n$. It is merely to express this differently to say that we can find a rational number lying on either side of r and differing from r by *as little as we please*. Again, given any two rational numbers r and s, we can interpolate between them a chain of rational numbers in which any two consecutive terms differ by as little as we please, that is to say by less than $1/n$, where n is any positive integer assigned beforehand.

From these considerations the reader might be tempted to infer that an adequate view of the nature of the line could be obtained by imagining it to be formed simply by the rational points which lie on it. And it is certainly the case that if we imagine the line to be made up solely of the rational points, and all other points (if there are any such) to be eliminated, the figure which remained would possess most of the properties which common sense attributes to the straight line, and would, to put the matter roughly, look and behave very much like a line.

A little further consideration, however, shows that this view would involve us in serious difficulties.

Let us look at the matter for a moment with the eye of common sense, and consider some of the properties which we may reasonably expect a straight line to possess if it is to satisfy the idea which we have formed of it in elementary geometry.

The straight line must be composed of points, and any segment of it by all the points which lie between its end points. With any such segment must be associated a certain entity called its *length*, which must be a *quantity* capable of *numerical measurement* in terms of any standard or unit length, and these lengths must be capable of combination with one another, according to the ordinary rules of algebra, by means of addition or multiplication.

Again, it must be possible to construct a line whose length is the sum or product of any two given lengths. If the length PQ, along a given line, is a, and the length QR, along the same straight line, is b, the length PR must be $a+b$. Moreover, if the lengths OP, OQ, along one straight line, are 1 and a, and the length OR along another straight line is b, and if we determine the length OS by Euclid's construction (Euc. VI. 12) for a fourth proportional to the lines OP, OQ, OR, this length must be ab, the algebraical fourth proportional to 1, a, b. And it is hardly necessary to remark that the sums and products thus defined must obey the ordinary 'laws of algebra'; viz.

$$a+b=b+a, \quad a+(b+c)=(a+b)+c,$$

$$ab=ba, \quad a(bc)=(ab)c, \quad a(b+c)=ab+ac.$$

The lengths of our lines must also obey a number of obvious laws concerning inequalities as well as equalities: thus if A, B, C are three points lying along Λ from left to right, we must have $AB<AC$, and so on. Moreover it must be possible, on our fundamental line Λ, to find a point P such that A_0P is equal to any segment whatever taken along Λ or along any other straight line. All these properties of a line, and more, are involved in the presuppositions of our elementary geometry.

Now it is very easy to see that the idea of a straight line as composed of a series of points, each corresponding to a rational number, cannot possibly satisfy all these requirements. There

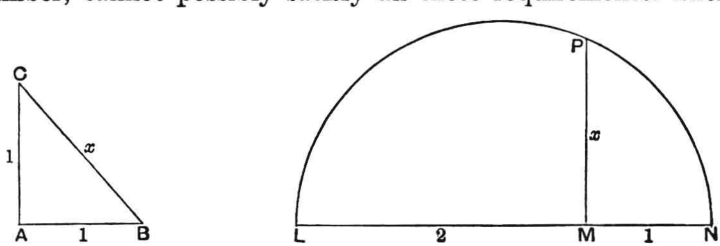

Fig. 2

are various elementary geometrical constructions, for example, which purport to construct a length x such that $x^2=2$. For instance, we may construct an isosceles right-angled triangle