ONE-PARAMETER SEMIGROUPS

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Preface

In the quarter century which has elapsed since the publication of the monumental treatise of Hille and Phillips (1957), the vitality of the theory of one-parameter semigroups has been clearly demonstrated in a variety of fields. Although many volumes have been published using one-parameter semigroups in various areas of applied mathematics, there seems to be no recent text giving a connected account of the abstract theory. This book is an attempt to rectify that situation.

One of the main problems in writing the book has been deciding how far to go into the applications. It would have been easy to double the length of the book by including some of these, and I could not have hoped to rival the specialist books even with such an expansion. In the end I have decided to avoid applications, and to choose examples not on the grounds of their importance, which is in any case a matter of judgement, but according to their value as illustrations of the general theory. It is nevertheless possible that my own interests in quantum theory and probability theory may be revealed in the examples in spite of my intentions.

The following is a short list of topics which I have not included in the book. Even this is not complete and I can only apologise for my ignorance in not mentioning books and even fields of study to which readers may have devoted large parts of their working lives.

1. One-parameter semigroups which are not of type c_0 are studied thoroughly by Hille and Phillips (1957), Krein (1971).
2. Clear and systematic accounts of the perturbation theory of point spectrum are given by Kato (1966) and Reed and Simon (1978).
3. For applications to probability and potential theory see Bery and Forst (1975), Dynkin (1965), Feller (1966), Loève (1955), Meyer (1966).
4. The general theory of spectral operators is covered by Colojoară and Foiaş (1968), Dowson (1977), Dunford and Schwartz (1971).
5. Applications to C^*-algebras and quantum statistical mechanics are given by Davies (1976*a*), Emch (1972), Evans and Lewis (1977), Pedersen (1979).

6. The theory of hypercontractive semigroups and its uses in quantum field theory are described in Reed and Simon (1975), Simon (1974).
7. For non-linear semigroups see Barbu (1976), Brezis (1973), Browder (1976), Goldstein (1972), Kato (1975*b*), Vainberg (1977).
8. For time-dependent evolution equations see Belleni-Morante (1979), Friedman (1969), Kato (1970), Krein (1971), Tanabe (1979).
9. For functional differential equations, control theory and approximation theory see Balakrishnan (1976), Butzer and Berens (1967), Hale (1977).

The first three chapters of the book, which form its core, should be accessible to anyone who has taken a thorough undergraduate course in functional analysis. However, the text is frequently illustrated with examples, many of which involve differential operators on L^p spaces and presuppose some familiarity with Lebesgue integration and Fourier analysis. These topics are also required for the main text of Chapters 7 and 8. Most of the material in Chapter 7 can be generalized to arbitrary Banach lattices, but I have chosen to present it only for L^p spaces in order to reduce the required knowledge.

Chapters 4 and 6 are devoted to special aspects of the theory for Hilbert spaces and for self-adjoint operators. I have assumed a familiarity with the spectral theorem for a single unitary operator, and then used this special case to prove the spectral theorem for any unbounded self-adjoint operator. The theory of quadratic forms is developed from first principles.

It is my great pleasure to thank the many people who have made the writing of this book possible. Among them I must mention D. A. Edwards, who introduced me to the subject many years ago, and also made a number of very helpful criticisms of the manuscript. I am grateful to the members of the Mathematical Institute and the Fellows of St. John's College for providing a happy and stimulating environment for academic research. Finally, but not least, I wish to thank Miss Onions for her very efficient typing of the manuscript.

E. B. Davies
October 1979

Contents

Chapter 1

Semigroups and Their Generators

1. BASIC PROPERTIES

We define a (jointly continuous, or c_0) *one-parameter semigroup* on a complex Banach space $\mathscr{B}$ to be a family T_t of bounded linear operators $T_t : \mathscr{B} \to \mathscr{B}$ parametrized by real $t \geq 0$ and satisfying the following relations:

(i) $T_0 = 1$.

(ii) If $0 \leq s, t < \infty$ then

$$T_s T_t = T_{s+t}.$$

(iii) The map

$$t, f \to T_t f \quad \text{from } [0, \infty) \times \mathscr{B} \text{ to } \mathscr{B} \text{ is jointly continuous.}$$

We shall discuss later the possibility of weakening the condition (iii).

In many areas of applied mathematics one-parameter semigroups arise in connection with the Cauchy problem for the differential equation

$$f_t' = Z f_t, \tag{1.1}$$

where Z is an operator on $\mathscr{B}$, and $'$ denotes derivative. Formally the solution of (1.1) is

$$f_t = T_t f_0,$$

where

$$T_t = e^{Zt} \tag{1.2}$$

satisfies (i) and (ii). The problem with these formal calculations is that in most applications Z is an unbounded operator, so the meaning of (1.2) is unclear. Much of the first two chapters of the book is devoted to a careful treatment of problems related to the unboundedness of Z.

The (infinitesimal) *generator* Z of a one-parameter semigroup T_t is defined by

$$Zf = \lim_{t\downarrow 0} t^{-1}(T_t f - f)$$

the *domain* $\mathrm{Dom}(Z)$ of Z being the set of f for which the limit exists. It is evident that $\mathrm{Dom}(Z)$ is a linear subspace of $\mathscr{B}$ and that Z is a linear operator from $\mathrm{Dom}(Z)$ into $\mathscr{B}$. It is not generally the case that $\mathrm{Dom}(Z)$ equals $\mathscr{B}$, but we shall prove that $\mathrm{Dom}(Z)$ is a dense subspace of $\mathscr{B}$.

Before doing this we need to discuss integration in Banach spaces. If $f:[a, b]\to\mathscr{B}$ is a continuous function, there is an element of $\mathscr{B}$, denoted by

$$\int_a^b f(x)\,\mathrm{d}x,$$

and defined by approximating f uniformly by piecewise constant functions, for which the definition of the integral is evident. It is easy to show that the integral depends linearly on f and that

$$\left\| \int_a^b f(x)\,\mathrm{d}x \right\| \leq \int_a^b \|f(x)\|\,\mathrm{d}x.$$

The integral may also be defined for suitable continuous functions $f:\mathbb{R}\to B$. Many other familiar results, such as the fundamental theorem of calculus, and the possibility of taking a bounded linear operator under the integral sign, may be proved by the traditional method.

LEMMA 1.1. *The subspace* $\mathrm{Dom}(Z)$ *is dense in* $\mathscr{B}$, *and invariant under* T_t *in the sense that*

$$T_t\{\mathrm{Dom}(Z)\} \subseteq \mathrm{Dom}(Z)$$

for all $t \geq 0$. *Moreover*

$$T_t Z f = Z T_t f$$

for all $f \in \mathrm{Dom}(Z)$ *and* $t \geq 0$.

Proof. If $f \in \mathscr{B}$ and

$$f_t = \int_0^t T_x f\,\mathrm{d}x$$

then

$$\begin{aligned}\lim_{h\downarrow 0} & h^{-1}(T_h f_t - f_t) \\ &= \lim_{h\downarrow 0}\left\{h^{-1}\int_h^{t+h} T_x f\,\mathrm{d}x - h^{-1}\int_0^t T_x f\,\mathrm{d}x\right\} \\ &= \lim_{h\downarrow 0}\left\{h^{-1}\int_t^{t+h} T_x f\,\mathrm{d}x - h^{-1}\int_0^h T_x f\,\mathrm{d}x\right\} \\ &= T_t f - f.\end{aligned}$$

Therefore $f_t \in \mathrm{Dom}(Z)$ and

$$Z(f_t) = T_t f - f. \tag{1.3}$$

Since $t^{-1}f_t \to f$ in norm as $t\downarrow 0$ we see that $\mathrm{Dom}(Z)$ is dense in $\mathscr{B}$.

If $f \in \mathrm{Dom}(Z)$ and $t \geqslant 0$ then

$$\begin{aligned}\lim_{h\downarrow 0} h^{-1}(T_h - 1)T_t f &= \lim_{h\downarrow 0} T_t\{h^{-1}(T_h - 1)f\} \\ &= T_t Z f.\end{aligned}$$

Hence $T_t f \in \mathrm{Dom}(Z)$ and $T_t Z f = Z T_t f$.

LEMMA 1.2. *If* $f \in \mathrm{Dom}(Z)$ *then*

$$T_t f - f = \int_0^t T_x Z f\,\mathrm{d}x.$$

Proof. If $f \in \mathrm{Dom}(Z)$ and ϕ lies in the Banach dual space $\mathscr{B}^*$ of $\mathscr{B}$, we define the complex-valued function $F(t)$ by

$$F(t) = \left\langle T_t f - f - \int_0^t T_x Z f\,\mathrm{d}x,\ \phi\right\rangle.$$

Its right hand derivative $D^+F(t)$ is given by

$$D^+F(t) = \langle Z T_t f - T_t Z f, \phi\rangle = 0.$$

Since $F(0) = 0$ and F is continuous, we see that $F(t) = 0$ for all $t \in [0, \infty)$. Since $\phi \in \mathscr{B}^*$ is arbitrary, the lemma follows by an application of the Hahn–Banach theorem.

LEMMA 1.3. *If* $f \in \mathrm{Dom}(Z)$ *then* $f_t = T_t f$ *is continuously differentiable on* $[0, \infty)$ *with*

$$f_t' = Z f_t.$$

Proof. The right-differentiability of $T_t f$ was established in Lemma 1.1. The left derivative at points $0 < t < \infty$ is given by

$$D^- T_t f = \lim_{h \downarrow 0} h^{-1}(T_t f - T_{t-h} f)$$

$$= \lim_{h \downarrow 0} h^{-1} \int_{t-h}^{t} T_x Zf \, dx,$$

by Lemma 1.2,

$$= T_t Zf = ZT_t f.$$

If Z is an operator with domain $\mathcal{D}$ in a Banach space $\mathcal{B}$ we say that Z is *closed* if $f_n \in \mathcal{D}$, $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} Zf_n = g$ together imply that $f \in \mathcal{D}$ and $Zf = g$. Equivalently if the *graph* of Z is defined by

$$\mathrm{Gr}(Z) = \{(f, g) \in \mathcal{B} \times \mathcal{B} : f \in \mathcal{D} \text{ and } Zf = g\}$$

then Z is closed if and only if $\mathrm{Gr}(Z)$ is a closed subspace of $\mathcal{B} \times \mathcal{B}$, which is given the *product norm*

$$\|(f, g)\| = \|f\| + \|g\|.$$

Problem 1.4. Use the Hahn–Banach theorem on $\mathcal{B} \times \mathcal{B}$ to show that if Z is closed then it is also weakly closed in the sense that if $f_n \in \mathrm{Dom}(Z)$ and

$$\operatorname*{w-lim}_{n\to\infty} f_n = f, \qquad \operatorname*{w-lim}_{n\to\infty} Zf_n = g$$

then $f \in \mathrm{Dom}(Z)$ and $Zf = g$. Note that we say that f_n *converges weakly* to f, or

$$\operatorname*{w-lim}_{n\to\infty} f_n = f$$

if

$$\lim_{n\to\infty} \langle f_n, \phi \rangle = \langle f, \phi \rangle$$

for all $\phi \in \mathcal{B}^*$.

LEMMA 1.5. *The generator Z of a one-parameter semigroup T_t is a closed operator.*

Proof. Suppose $f_n \in \mathrm{Dom}(Z)$, $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} Zf_n = g$. Then using Lemma 1.2 we obtain

$$\begin{aligned} T_t f - f &= \lim_{n\to\infty} (T_t f_n - f_n) \\ &= \lim_{n\to\infty} \int_0^t T_x Z f_n \,\mathrm{d}x \\ &= \int_0^t T_x g \,\mathrm{d}x. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{t\downarrow 0} t^{-1}(T_t f - f) &= \lim_{t\downarrow 0} t^{-1} \int_0^t T_x g \,\mathrm{d}x \\ &= g, \end{aligned}$$

so $f \in \mathrm{Dom}(Z)$ and $Zf = g$.

LEMMA 1.6. *The space* $\mathrm{Dom}(Z)$ *is complete with respect to the norm*

$$|||f||| = \|f\| + \|Zf\|. \tag{1.4}$$

Moreover T_t *is a one-parameter semigroup on* $\mathrm{Dom}(Z)$ *for this norm.*

Proof. The map $j:\mathrm{Dom}(Z) \to \mathcal{B} \times \mathcal{B}$ defined by $j(f) = (f, Zf)$ is one–one with range equal to $\mathrm{Gr}(Z)$. The first statement of the lemma follows from the fact that $\mathrm{Gr}(Z)$ is closed and

$$|||f||| = \|j(f)\|.$$

The restriction of T_t to $\mathrm{Dom}(Z)$ is bounded by Lemma 1.1 and trivially satisfies conditions (i) and (ii). To prove (iii) we note that if $\tilde{T}_t$ is defined on $\mathcal{B} \times \mathcal{B}$ by

$$\tilde{T}_t(f, g) = (T_t f, T_t g)$$

then $\tilde{T}_t$ satisfies (i)–(iii) and

$$j(T_t f) = \tilde{T}_t j(f)$$

for all $f \in \mathrm{Dom}(Z)$.

THEOREM 1.7. *Let* Z *be the generator of a one-parameter semigroup* T_t. *If a function* $f:[0, a] \to \mathrm{Dom}(Z)$ *satisfies*

$$f_t' = Zf_t \tag{1.5}$$

for all $t \in [0, a]$, *then*

$$f_a = T_a f_0. \tag{1.6}$$

Hence T_t *is uniquely determined by* Z.

Proof. Given f_t and $\phi \in \mathscr{B}^*$, define

$$F(t) = \langle T_t f_{a-t}, \phi \rangle.$$

Then

$$\begin{aligned} D^+F(t) &= \lim_{h \downarrow 0} \langle h^{-1}\{T_{t+h} f_{a-t-h} - T_t f_{a-t}\}, \phi \rangle \\ &= \lim_{h \downarrow 0} \langle T_{t+h} h^{-1}\{f_{a-t-h} - f_{a-t}\}, \phi \rangle \\ &\quad + \lim_{h \downarrow 0} \langle h^{-1}\{T_{t+h} - T_t\} f_{a-t}, \phi \rangle \\ &= -\langle T_t Z f_{a-t}, \phi \rangle + \langle Z T_t f_{a-t}, \phi \rangle \\ &= 0. \end{aligned}$$

Since F is continuous it is constant, and

$$\langle T_a f_0, \phi \rangle = \langle f_a, \phi \rangle$$

for all $\phi \in \mathscr{B}^*$. This implies (1.6).

By Lemma 1.3 and Theorem 1.7 the Cauchy problem for the differential equation

$$f_t' = Z f_t$$

(i.e. the existence of a solution for arbitrary initial data) is uniquely soluble if Z is the generator of a one-parameter semigroup and $f_0 \in \mathrm{Dom}(Z)$, with solution which depends continuously on f_0. Since the solutions determine the semigroup we shall often write

$$T_t = e^{Zt}$$

below, without suggesting that the right hand side is more than a formal expression.

The converse problems of determining which operators Z are generators of one-parameter semigroups, and for which operators Z the Cauchy problem is soluble, are not trivial, and the first of them occupies much of Chapter 2.

In applications the domain of a generator Z is often rather complicated to describe, and one therefore prefers to work in a slightly smaller subspace $\mathscr{D}$. We say that $\mathscr{D} \subseteq \mathrm{Dom}(Z)$ is a *core* for an arbitrary closed operator Z if for all $f \in \mathrm{Dom}(Z)$ there exists a sequence $f_n \in \mathscr{D}$ such that

$$\lim_{n\to\infty} f_n = f, \qquad \lim_{n\to\infty} Zf_n = Zf.$$

Equivalently $\mathscr{D}$ is a core for Z if it is dense in $\mathrm{Dom}(Z)$ for the norm defined in Lemma 1.6.

If Z and Y are two operators on a Banach space we say that Y is an *extension* of Z if $\mathrm{Dom}(Z) \subseteq \mathrm{Dom}(Y)$ and $Zf = Yf$ for all $f \in \mathrm{Dom}(Z)$. We say that Z is *closable* if it has a closed extension.

LEMMA 1.8. *The operator Z is closable if and only if $f_n \in \mathrm{Dom}(Z)$, $\lim_{n\to\infty} f_n = 0$ and $\lim_{n\to\infty} Zf_n = g$ imply that $g = 0$. If Z is closable then it has a least closed extension, called its closure.*

Proof. We first note that Y is an extension of Z if and only if

$$\mathrm{Gr}(Z) \subseteq \mathrm{Gr}(Y)$$

and that Y is closed if and only if $\mathrm{Gr}(Y)$ is a closed subspace of $\mathscr{B} \times \mathscr{B}$. When this happens we show that the closure L of $\mathrm{Gr}(Z)$ in $\mathscr{B} \times \mathscr{B}$ is a subspace which is the graph of an operator X. For L to be a graph it is only necessary to show that if $(f, g_1) \in L$ and $(f, g_2) \in L$ then $g_1 = g_2$. But if $(f, g_i) \in L$ then $(f, g_i) \in \mathrm{Gr}(Y)$ so $g_i = Yf$ and $g_1 = g_2$. It is obvious that X is the least closed extension of Z.

If Z has a closed extension Y and $f_n \in \mathrm{Dom}(Z)$, $\lim_{n\to\infty} f_n = 0$ and $\lim_{n\to\infty} Zf_n = g$, then since Y is closed $0 \in \mathrm{Dom}(Y)$ and $Y(0) = g$. Since Y is linear $g = 0$.

Conversely, suppose Z has no closed extension. Then the closure L of $\mathrm{Gr}(Z)$ is not the graph of any operator, so there exist $(f, g_1) \in L$ and $(f, g_2) \in L$ with $g_1 \neq g_2$. There exist sequences f_n^1, f_n^2 in $\mathrm{Dom}(Z)$ such that

$$\lim_{n\to\infty} f_n^1 = f, \qquad \lim_{n\to\infty} Zf_n^1 = g_1,$$

$$\lim_{n\to\infty} f_n^2 = f, \qquad \lim_{n\to\infty} Zf_n^2 = g_2.$$

Clearly if $f_n = f_n^1 - f_n^2$ then

$$\lim_{n\to\infty} f_n = 0, \qquad \lim_{n\to\infty} Zf_n = g_1 - g_2 \neq 0.$$

It is often not easy to determine whether a given subspace $\mathscr{D}$ of $\mathrm{Dom}(Z)$ is a core, but the following criterion is useful when the semigroup T_t is explicitly given.

THEOREM 1.9. *If $\mathscr{D} \subseteq \mathrm{Dom}(Z)$ is dense in $\mathscr{B}$ and invariant under the semigroup T_t, then $\mathscr{D}$ is a core for Z.*

Proof. We use Lemma 1.6 and work simultaneously with the two norms. Let $\bar{\mathscr{D}}$ denote the closure of $\mathscr{D}$ in $\mathrm{Dom}(Z)$ with respect to $|||\cdot|||$. If $f \in \mathrm{Dom}(Z)$ then by the density of $\mathscr{D}$ in $\mathscr{B}$ there is a sequence $f_n \in \mathscr{D}$ such that $\|f_n - f\| \to 0$. Since $t \to T_t f_n$ is continuous for the $|||\cdot|||$ metric we have

$$\int_0^t T_x f_n \, \mathrm{d}x \in \bar{\mathscr{D}}.$$

By (1.3)

$$\begin{aligned}\lim_{n\to\infty} \left|\left|\left| \int_0^t T_x f_n \, \mathrm{d}x - \int_0^t T_x f \, \mathrm{d}x \right|\right|\right| &= \lim_{n\to\infty} \left\| \int_0^t T_x (f_n - f) \, \mathrm{d}x \right\| \\ &\quad + \lim_{n\to\infty} \| T_t f_n - f_n - T_t f + f\| \\ &= 0,\end{aligned}$$

so

$$\int_0^t T_x f \, \mathrm{d}x \in \bar{\mathscr{D}}.$$

By (1.3) once again,

$$\begin{aligned}\lim_{t\downarrow 0} \left|\left|\left| t^{-1} \int_0^t T_x f \, \mathrm{d}x - f \right|\right|\right| &= \lim_{t\downarrow 0} \left\| t^{-1} \int_0^t T_x f \, \mathrm{d}x - f \right\| \\ &\quad + \lim_{t\downarrow 0} \|t^{-1}(T_t f - f) - Zf\| \\ &= 0,\end{aligned}$$

so $f \in \bar{\mathscr{D}}$. This proves that $\bar{\mathscr{D}} = \mathrm{Dom}(Z)$ as required.

Example 1.10. Let $L^p(\mathbb{R}^n)$ denote the Banach space of complex-valued measurable functions f on $\mathbb{R}^n$ with finite norm

$$\|f\| = \left\{ \int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}^n x \right\}^{1/p},$$