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Topological Modular Forms

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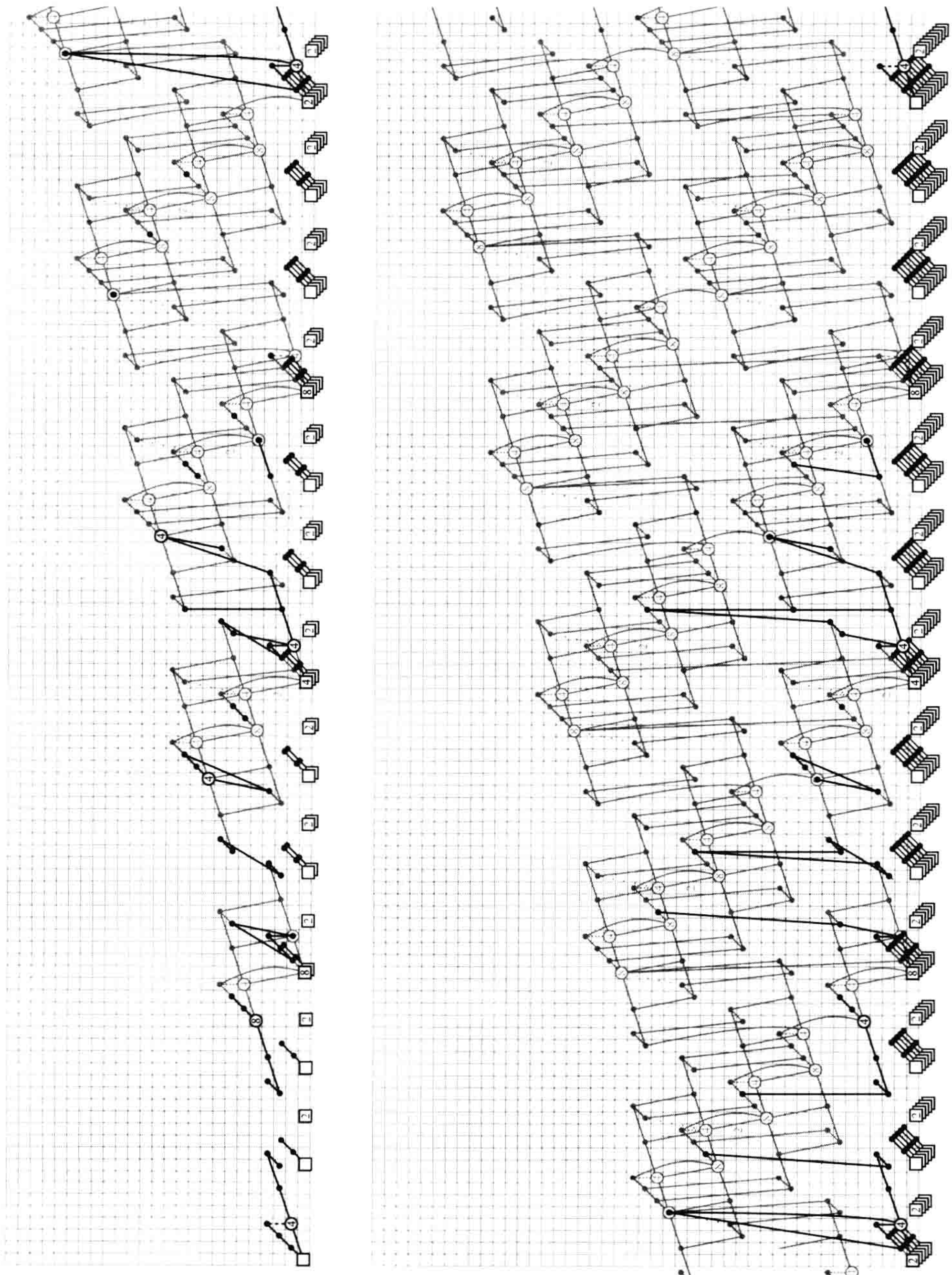


Preface

This is a book about the theory of topological modular forms. It is also a record of the efforts of a group of graduate students to learn that theory at the 2007 Talbot Workshop, and so a book born of and steeped in the Talbot vision.

In the fall of 2003, Mike Hopkins taught a course at MIT about *tmf*. Our generation of Cambridge algebraic topologists, having survived and thrived in Haynes Miller's Kan seminar, found in Mike's class our next, and really our last common, mathematical crucible. The course hacked through the theory of algebraic modular forms, formal groups, multiplicative stable homotopy theory, stacks, even more stacks, moduli stacks of elliptic curves, Bousfield localization, Morava K- and E-theory, the arithmetic and Hasse squares, André–Quillen cohomology, obstruction theory for moduli of associative and commutative ring spectra—by this point we were having dreams, or maybe nightmares, about the spiral exact sequence.

In the middle of the course, we all flew over to Münster for a week-long workshop on *tmf* with lectures by Mike, Haynes, Matt Ando, Charles Rezk, and Paul Goerss. A transatlantic mix of students spent the late afternoons coaxing and cramming the knowledge in at a cafe off Steinfurter Strasse; there we devised a plan to reconvene and sketched a vision of what would become the Talbot Workshops: a gathering for graduate students, focused on a single topic of contemporary research interest, lectured by graduate students and guided by a single faculty mentor, having talks in the morning and in the evening and every afternoon free for discussion and outdoor activities, with participants sleeping and lecturing and cooking together under the same roof. We pitched it to Mike and Haynes and they agreed to back a ragtag summit. Talbot was born.

Three years later, in 2007, we decided to bring Talbot home with a workshop on *tmf*, mentored by Mike Hopkins. Mike stopped by Staples on his way to the workshop and picked up a big red "That was easy" button. Throughout the workshop, whenever he or anyone else completed a particularly epic spectral sequence computation or stacky decomposition, he'd hit the button and a scratchy electronic voice would remind us, "That was easy!" It became the workshop joke (for much of it was evidently not easy) and mantra (for shifting perspective, whether to multiplicative stable homotopy or to stacky language or to a suitable localization, did make the intractable seem possible).

This book is a record and expansion of the material covered in the Talbot 2007 workshop. Though the authors of the various chapters have brought their own expositional perspectives to bear (particularly heroically in the case of Mark Behrens), the contemporary material in this book is due to Mike Hopkins, Haynes Miller, and Paul Goerss, with contributions by Mark Mahowald, Matt Ando, and Charles Rezk.

Acknowledgments

We thank the participants of Talbot 2007 for their dedication and enthusiasm during the workshop: Ricardo Andrade, Vigleik Angeltveit, Tilman Bauer, Mark Behrens, Thomas Bitoun, Andrew Blumberg, Ulrik Buchholtz, Scott Carnahan, John Duncan, Matthew Gelvin, Teena Gerhardt, Veronique Godin, Owen Gwilliam, Henning Hohnhold, Valentina Joukhovitski, Jacob Lurie, Carl Mautner, Justin Noel, Corbett Redden, Nick Rozenblyum, and Samuel Wüthrich. Special thanks to Corbett, Carl, Henning, Tilman, Jacob, and Vigleik for writing up their talks from the workshop. And super-special thanks to Mark for his enormous effort writing up his perspective on the construction of *tmf*, and for extensive support and assistance throughout the development of the book. Our appreciation and thanks also to Tilman, to Mark, and to Niko Naumann for substantial contributions to and help with the glossary, and to Paul Goerss, Charles Rezk, and David Gepner for suggestions and corrections.

Our appreciation goes to Nora Ganter, who in 2003 ran a student seminar on *tmf*—it was our first sustained exposure to the subject; Nora also prepared the original literature list for the Talbot 2007 program. We also acknowledge the students who wrote up the 'Course notes for elliptic cohomology' based on Mike Hopkins' 1995 course, and the students who wrote up 'Complex oriented cohomology theories and the language of stacks' based on Mike's 1999 course—both documents, distributed through the topology underground, were helpful for us in the years leading up to and at Talbot 2007.

The book would not have happened without Talbot, and the Talbot workshops would not have happened without the support of MIT and the NSF. Christopher Stark at the NSF was instrumental in us securing the first Talbot grant, DMS-0512714, which funded the workshops from 2005 til 2008. MIT provided the facilities for the first workshop, at the university retreat Talbot House, and has provided continual logistical, administrative, and technical support for the workshops ever since. We'd like to thank the second and third generations of Talbot organizers, Owen Gwilliam, Sheel Ganatra, and Hiro Lee Tanaka, and Saul Glasman, Gijs Heuts, and Dylan Wilson, for carrying on the tradition.

During the preparation of this book, we were supported by grants and fellowships from the NSF, the Miller Institute, and the EPSRC. MSRI, as part of a semester program on Algebraic Topology in the spring of 2014, provided an ideal working environment and support for the completion of the book. Our great thanks and appreciation to our editor, Sergei Gelfand, who shepherded the book along and without whose enthusiasm, encouragement, and pressure, the book might have taken another seven years.

Our greatest debt is to Haynes Miller and Mike Hopkins. Haynes was and has been a source of wisdom and insight and support for us through the years, and it is with fondness and appreciation that we acknowledge his crucial role in our mathematical upbringing and thus in the possibility of this book. We are singularly grateful to our advisor, Mike, for inspiring us and guiding us, all these years, with his characteristic energy and brilliance, humor and care. Thank you.

Introduction

CONTENTS

1. Elliptic cohomology

A ring-valued cohomology theory E is *complex orientable* if there is an 'orientation class' $x \in E^2(\mathbb{CP}^\infty)$ whose restriction along the inclusion $S^2 \cong \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ is the element 1 in $E^0S^0 \cong E^2\mathbb{CP}^1$. The existence of such an orientation class implies, by the collapse of the Atiyah–Hirzebruch spectral sequence, that

$$E^*(\mathbb{CP}^\infty) \cong E^*[[x]].$$

The class x is a universal characteristic class for line bundles in E-cohomology; it is the E-theoretic analogue of the first Chern class. The space $\mathbb{CP}^\infty$ represents the functor

$$X \mapsto \{\text{isomorphism classes of line bundles on } X\},$$

and the tensor product of line bundles induces a multiplication map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$. Applying E^* produces a ring map

$$E^*[[x]] \cong E^*(\mathbb{CP}^\infty) \to E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E^*[[x_1, x_2]];$$

the image of x under this map is a formula for the E-theoretic first Chern class of a tensor product of line bundles in terms of the first Chern classes of the two factors. That ring map $E^*[[x]] \to E^*[[x_1, x_2]]$ is a (1-dimensional, commutative) formal group law—that is, a commutative group structure on the formal completion $\hat{\mathbb{A}}^1$ at the origin of the affine line $\mathbb{A}^1$ over the ring E^*.

A formal group often arises as the completion of a group scheme at its identity element; the dimension of the formal group is the dimension of the original group scheme. There are three kinds of 1-dimensional group schemes:

(1) the additive group $\mathbb{G}_a = \mathbb{A}^1$ with multiplication determined by the map $\mathbb{Z}[x] \to \mathbb{Z}[x_1, x_2]$ sending x to $x_1 + x_2$,
(2) the multiplicative group $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$ with multiplication determined by the map $\mathbb{Z}[x^{\pm 1}] \to \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ sending x to $x_1 x_2$, and
(3) elliptic curves (of which there are many isomorphism classes).

Ordinary cohomology is complex orientable, and its associated formal group is the formal completion of the additive formal group. Topological K-theory is also complex orientable, and its formal group is the formal completion of the multiplicative formal group. This situation naturally leads one to search for 'elliptic' cohomology theories whose formal groups are the formal completions of elliptic curves. These elliptic cohomology theories should, ideally, be functorial for morphisms of elliptic curves.

Complex bordism MU is complex orientable and the resulting formal group law is the universal formal group law; this means that ring maps from MU_* to R are in natural bijective correspondence with formal group laws over R. Given a commutative ring R and a map $MU_* \to R$ that classifies a formal group law over R, the functor

$$X \mapsto MU_*(X) \otimes_{MU_*} R$$

is a homology theory if and only if the corresponding map from $\mathrm{Spec}(R)$ to the moduli stack $\mathcal{M}_{FG}$ of formal groups is flat. There is a map

$$\mathcal{M}_{ell} \to \mathcal{M}_{FG}$$

from the moduli stack of elliptic curves to that of formal groups, sending an elliptic curve to its completion at the identity; this map is flat. Any flat map $\mathrm{Spec}(R) \to \mathcal{M}_{ell}$ therefore provides a flat map $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$ and thus a homology theory, or equivalently, a cohomology theory (a priori only defined on finite CW-complexes). In other words, to any affine scheme with a flat map to the moduli stack of elliptic curves, there is a functorially associated cohomology theory.

The main theorem of Goerss–Hopkins–Miller is that this functor (that is, presheaf)

$$\{\text{flat maps from affine schemes to } \mathcal{M}_{ell}\} \to \{\text{multiplicative cohomology theories}\},$$

when restricted to maps that are étale, lifts to a sheaf

$$\mathcal{O}^{\mathrm{top}}: \ \{\text{étale maps to } \mathcal{M}_{ell}\} \to \{E_\infty\text{-ring spectra}\}.$$

(Here the subscript 'top' refers to it being a kind of 'topological', rather than discrete, structure sheaf.) The value of this sheaf on $\mathcal{M}_{ell}$ itself, that is the E_∞-ring spectrum of global sections, is the periodic version of the spectrum of topological modular forms:

$$TMF := \mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell}) = \Gamma(\mathcal{M}_{ell}, \mathcal{O}^{\mathrm{top}}).$$

The spectrum TMF owes its name to the fact that its ring of homotopy groups is rationally isomorphic to the ring

$$\mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 - 1728\Delta) \ \cong \ \bigoplus_{n \geq 0} \Gamma\big(\mathcal{M}_{ell}, \omega^{\otimes n}\big)$$

of weakly holomorphic integral modular forms. Here, the elements c_4, c_6, and Δ have degrees 8, 12, and 24 respectively, and ω is the sheaf of invariant differentials (the restriction to $\mathcal{M}_{ell}$ of the (vertical) cotangent bundle of the universal elliptic curve $\mathcal{E} \to \mathcal{M}_{ell}$). That ring of modular forms is periodic with period 24, and the periodicity is given by multiplication by the discriminant Δ. The discriminant is not an element in the homotopy groups of TMF, but its twenty-fourth power $\Delta^{24} \in \pi_{24^2}(TMF)$ is, and, as a result, $\pi_*(TMF)$ has a periodicity of order $24^2 = 576$.

One would like an analogous E_∞-ring spectrum whose homotopy groups are rationally isomorphic to the subring

$$\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$$

of integral modular forms. For that, one observes that the sheaf $\mathcal{O}^{\mathrm{top}}$ is defined not only on the moduli stack of elliptic curves, but also on the Deligne–Mumford compactification $\overline{\mathcal{M}}_{ell}$ of the moduli stack—this compactification is the moduli stack of elliptic curves possibly with nodal singularities. The spectrum of global sections over $\overline{\mathcal{M}}_{ell}$ is denoted

$$Tmf := \mathcal{O}^{\mathrm{top}}(\overline{\mathcal{M}}_{ell}) = \Gamma(\overline{\mathcal{M}}_{ell}, \mathcal{O}^{\mathrm{top}}).$$

The element $\Delta^{24} \in \pi_{24^2}(Tmf)$ is no longer invertible in the homotopy ring, and so the spectrum *Tmf* is not periodic. This spectrum is not connective either, and the mixed capitalization reflects its intermediate state between the periodic version *TMF* and the connective version *tmf*, described below, of topological modular forms.

In positive degrees, the homotopy groups of *Tmf* are rationally isomorphic to the ring $\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$. The homotopy groups $\pi_{-1}, \ldots, \pi_{-20}$ are all zero, and the remaining negative homotopy groups are given by:

$$\pi_{-n}(Tmf) \cong \left[\pi_{n-21}(Tmf)\right]_{\textit{torsion-free}} \oplus \left[\pi_{n-22}(Tmf)\right]_{\textit{torsion}}.$$

This structure in the homotopy groups is a kind of Serre duality reflecting the properness (compactness) of the moduli stack $\overline{\mathcal{M}}_{ell}$.

If we take the (-1)-connected cover of the spectrum *Tmf*, that is, if we kill all its negative homotopy groups, then we get

$$tmf := Tmf\langle 0\rangle,$$

the connective version of the spectrum of topological modular forms. This spectrum is now, as desired, a topological refinement of the classical ring of integral modular forms. Note that one can recover *TMF* from either of the other versions by inverting the element Δ^{24} in the 576^{th} homotopy group:

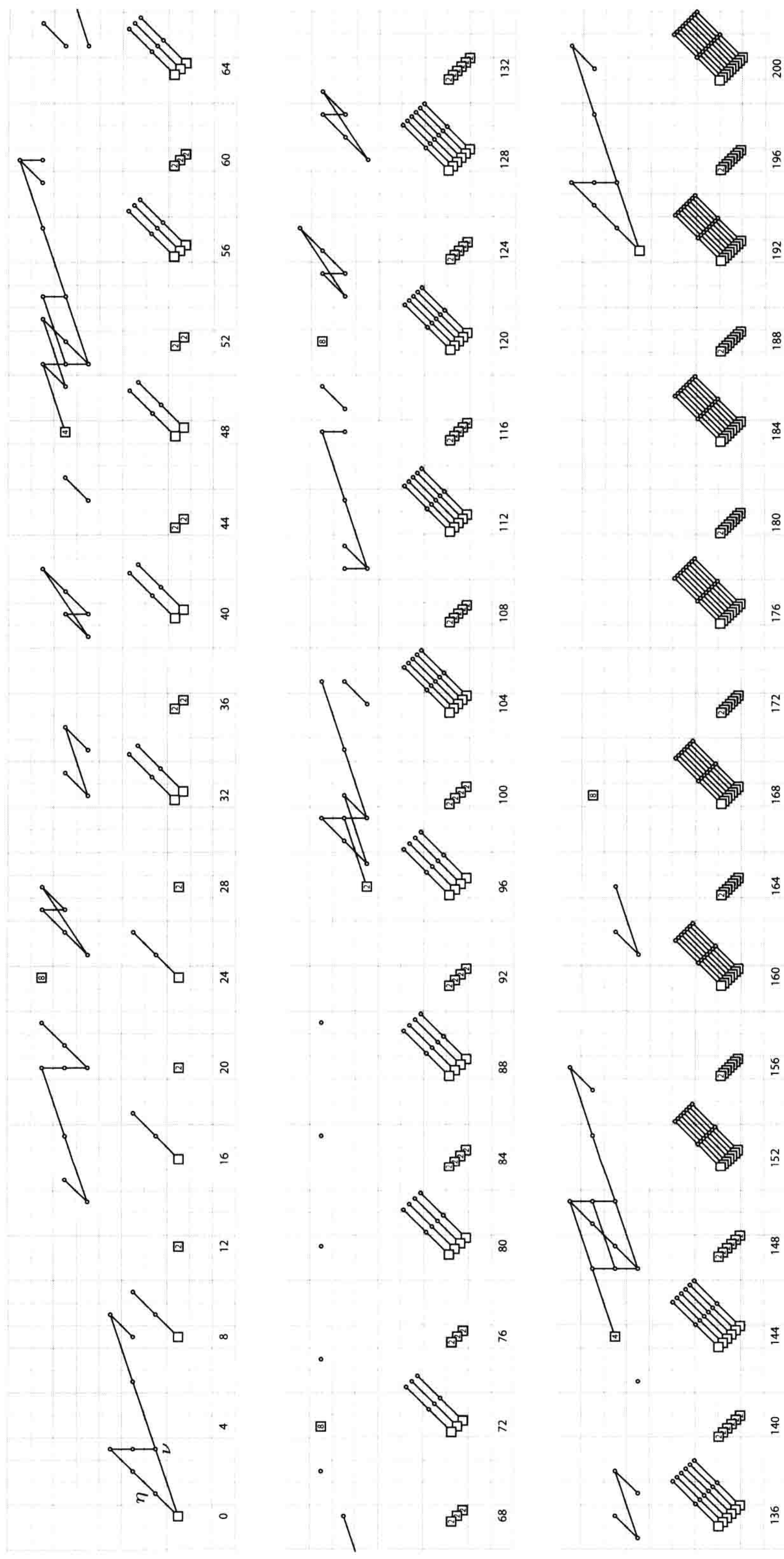
$$TMF = tmf[\Delta^{-24}] = Tmf[\Delta^{-24}].$$

There is another moduli stack worth mentioning here, the stack $\overline{\mathcal{M}}^{+}_{ell}$ of elliptic curves with possibly nodal or cuspidal singularities. There does not seem to be an extension of $\mathcal{O}^{\mathrm{top}}$ to that stack. However, if there were one, then a formal computation, namely an elliptic spectral sequence for that hypothetical sheaf, shows that the global sections of the sheaf over $\overline{\mathcal{M}}^{+}_{ell}$ would be the spectrum *tmf*. That hypothetical spectral sequence is the picture that appears before the preface. It is also, more concretely, the Adams–Novikov spectral sequence for the spectrum *tmf*.

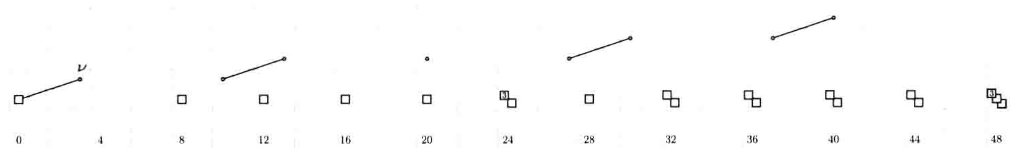
So far, we have only mentioned the connection between *tmf* and modular forms. The connection of *tmf* to the stable homotopy groups of spheres is equally strong and the unit map from the sphere spectrum to *tmf* detects an astounding amount of the 2- and 3-primary parts of the homotopy $\pi_*(\mathbb{S})$ of the sphere.

The homotopy groups of *tmf* are as follows at the prime 2:

The homotopy groups of tmf at the prime 2.



and as follows at the prime 3:



Here, a square indicates a copy of $\mathbb{Z}$ and a dot indicates a copy of $\mathbb{Z}/p$. A little number n drawn in a square indicates that the copy of $\mathbb{Z}$ in $\pi_*(tmf)$ maps onto an index n subgroup of the corresponding $\mathbb{Z}$ in the ring of modular forms. A vertical line between two dots indicates an additive extension, and a slanted line indicates the multiplicative action of the generator $\eta \in \pi_1(tmf)$ or $\nu \in \pi_3(tmf)$. The y-coordinate, although vaguely reminiscent of the filtration degree in the Adams spectral sequence, has no meaning in the above charts.

Note that, at the prime 2, the pattern on the top of the chart (that is, above the expanding *ko* pattern on the base) repeats with a periodicity of $192 = 8 \cdot 24$. A similar periodicity (not visible in the above chart) happens at the prime 3, with period $72 = 3 \cdot 24$. Over $\mathbb{Z}$, taking the least common multiple of these two periodicities results in a periodicity of $24 \cdot 24 = 576$.

2. A brief history of *tmf*

In the sixties, Conner and Floyd proved that complex K-theory is determined by complex cobordism: if X is a space, then its K-homology can be described as $K_*(X) \cong MU_*(X) \otimes_{MU_*} K_*$, where K_* is a module over the complex cobordism ring of the point via the Todd genus map $MU_* \to K_*$. Following this observation, it was natural to look for other homology theories that could be obtained from complex cobordism by a similar tensor product construction. By Quillen's theorem (1969), MU_* is the base ring over which the universal formal group law is defined; ring maps $MU_* \to R$ thus classify formal groups laws over R.

Given such a map, there is no guarantee in general that the functor $X \mapsto MU_*(X) \otimes_{MU_*} R$ will be a homology theory. If R is a flat MU_*-module, then long exact sequences remain exact after tensoring with R and so the functor in question does indeed define a new homology theory. However, the condition of being flat over MU_* is quite restrictive. Landweber's theorem (1976) showed that, because arbitrary MU_*-modules do not occur as the MU-homology of spaces, the flatness condition can be greatly relaxed. A more general condition, Landweber exactness, suffices to ensure that the functor $MU_*(-) \otimes_{MU_*} R$ satisfies the axioms of a homology theory. Shortly after the announcement of Landweber's result, Morava applied that theorem to the formal groups of certain elliptic curves and constructed the first elliptic cohomology theories (though the term 'elliptic cohomology' was coined only much later).

In the mid-eighties, Ochanine introduced certain genera (that is homomorphisms out of a bordism ring) related to elliptic integrals, and Witten constructed a genus that took values in the ring of modular forms, provided the low-dimensional characteristic classes of the manifold vanish. Landweber–Ravenel–Stong made explicit the connection between elliptic genera, modular forms, and elliptic cohomology by identifying the target of the universal Ochanine elliptic genus with the coefficient ring of the homology theory $X \mapsto MU_*(X) \otimes_{MU_*} \mathbb{Z}[\frac{1}{2}][\delta, \epsilon, \Delta^{-1}]$ associated to the Jacobi quartic elliptic curve $y^2 = 1 - 2\delta x^2 + \epsilon x^4$ (here, Δ is the discriminant of the polynomial in x). Segal had also presented a picture of the relationship between elliptic cohomology and Witten's physics-inspired index theory on loop spaces. In hindsight, a natural question would have been whether there existed a form of elliptic cohomology that received Witten's genus, thus explaining its integrality and modularity properties. But at the time, the community's attention was on Witten's rigidity conjecture for elliptic genera (established by Bott and Taubes), and on finding a geometric interpretation for elliptic cohomology—a problem that remains open to this day, despite a tantalizing proposal by Segal and much subsequent work.

Around 1989, inspired in part by work of McClure and Baker on A_∞ structures and actions on spectra and by Ravenel's work on the odd primary Arf invariant, Hopkins and Miller showed that a certain profinite group known as the Morava stabilizer group acts by A_∞ automorphisms on the Lubin–Tate spectrum E_n (the representing spectrum for the Landweber exact homology theory associated to the universal deformation of a height n formal group law). Of special interest was the action of the binary tetrahedral group on the spectrum E_2 at the prime 2. The homotopy fixed point spectrum of this action was called EO_2, by analogy with the real K-theory spectrum KO being the homotopy fixed points of complex conjugation on the complex K-theory spectrum.

Mahowald recognized the homotopy of EO_2 as a periodic version of a hypothetical spectrum with mod 2 cohomology $A/\!/A(2)$, the quotient of the Steenrod algebra by the submodule generated by Sq^1, Sq^2, and Sq^4. It seemed likely that there would be a corresponding connective spectrum eo_2 and indeed a bit later Hopkins and Mahowald produced such a spectrum; (in hindsight, that spectrum eo_2 is seen as the 2-localization of *tmf*). However, Davis–Mahowald (1982) had proved, by an intricate spectral sequence argument, that it is impossible to realize $A/\!/A(2)$ as the cohomology of a spectrum. This conundrum was resolved only much later, when Mahowald found a missing differential around the 55^{th} stem of the Adams spectral sequence for the sphere, invalidating the earlier Davis–Mahowald argument.

In the meantime, computations of the cohomology of $MO\langle 8\rangle$ at the prime 2 revealed an $A/\!/A(2)$ summand, suggesting the existence of a map of spectra from $MO\langle 8\rangle$ to eo_2. While attempting to construct a map $MO\langle 8\rangle \to EO_2$, Hopkins (1994) thought to view the binary tetrahedral group as the automorphism group of the supersingular elliptic curve at the prime 2; the idea of a sheaf of ring spectra over the moduli stack of elliptic curves quickly followed—the global sections of that sheaf, *TMF*, would then be an integral version of EO_2.

The language of stacks, initially brought to bear on complex cobordism and formal groups by Strickland, proved crucial for even formulating the question *TMF* would answer. In particular, the stacky perspective allowed a reformulation of Landweber's exactness criterion in a more conceptual and geometric way: $MU_* \to R$ is Landweber exact if and only if the corresponding map to the moduli stack of formal groups, $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$, is flat. From this viewpoint, Landweber's theorem defined a presheaf of homology theories on the flat site of the moduli stack $\mathcal{M}_{FG}$ of formal groups. Restricting to those formal groups coming from elliptic curves then provided a presheaf of homology theories on the moduli stack of elliptic curves.

Hopkins and Miller conceived of the problem as lifting this presheaf of homology theories to a sheaf of spectra. In the 80s and early 90s, Dwyer, Kan, Smith, and Stover had developed an obstruction theory for rigidifying a diagram in a homotopy category (here a diagram of elliptic homology theories) to an honest diagram (here a sheaf of spectra). Hopkins and Miller adapted the Dwyer–Kan–Stover theory to treat the seemingly more difficult problem of rigidifying a diagram of multiplicative cohomology theories to a diagram of A_∞-ring spectra. The resulting multiplicative obstruction groups vanished, except at the prime 2—Hopkins addressed that last case by a direct construction in the category of $K(1)$-local E_∞-ring spectra. Altogether the resulting sheaf of spectra provided a universal elliptic cohomology theory, the spectrum *TMF* of global sections (and its connective version *tmf*). Subsequently, Goerss and Hopkins upgraded the A_∞ obstruction theory to an obstruction theory for E_∞-ring spectra, leading to the definitive theorem of Goerss–Hopkins–Miller: the presheaf of elliptic homology theories on the compactified moduli stack of elliptic curves lifts to a sheaf of E_∞-ring spectra.

Meanwhile, Ando–Hopkins–Strickland (2001) established a systematic connection between elliptic cohomology and elliptic genera by constructing, for every elliptic cohomology theory E, an E-orientation for almost complex manifolds with certain vanishing characteristic classes. This collection was expected to assemble into a single unified multiplicative *tmf*-orientation. Subsequently Laures (2004) built a $K(1)$-local E_∞-map $MO\langle 8\rangle \to tmf$ and then finally Ando–Hopkins–Rezk produced the expected integral map of E_∞-ring spectra $MO\langle 8\rangle \to tmf$ that recovers Witten's genus at the level of homotopy groups.

Later, an interpretation of *tmf* was given by Lurie (2009) using the theory of spectral algebraic geometry, based on work of Töen and Vezzosi. Lurie interpreted the stack $\mathcal{M}_{ell}$ with its sheaf $\mathcal{O}^{\mathrm{top}}$ as a stack not over commutative rings but over E_∞-ring spectra. Using Goerss–Hopkins–Miller obstruction theory and a spectral form of Artin's representability theorem, he identified that stack as classifying oriented elliptic curves over E_∞-ring spectra. Unlike the previous construction of *tmf* and of the sheaf $\mathcal{O}^{\mathrm{top}}$, this description specifies the sheaf and therefore the spectrum *tmf* up to a contractible space of choices.

3. Overview

Part I

<u>Chapter 1</u>: **Elliptic genera and elliptic cohomology.** One-dimensional formal group laws entered algebraic topology though complex orientations, in answering the question of which generalized cohomology theories E carry a theory of Chern classes for complex vector bundles. In any such theory, the E-cohomology of $\mathbb{CP}^\infty$ is isomorphic to $E^*[[c_1]]$, the E-cohomology ring of a point adjoin a formal power series generator in degree 2. The tensor product of line bundles defines a map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, which in turn defines a comultiplication on $E^*[[c_1]]$, i.e., a formal group law. Ordinary homology is an example of such a theory; the associated formal group is the additive formal group, since the first Chern class of the tensor product of line bundles is the sum of the respective Chern classes, $c_1(L \otimes L') = c_1(L) + c_1(L')$. Complex K-theory is another example of such a theory; the associated formal group is the multiplicative formal group.

Complex cobordism also admits a theory of Chern classes, hence a formal group. Quillen's theorem is that this is the universal formal group. In other words, the formal group of complex cobordism defines a natural isomorphism of MU^* with the Lazard ring, the classifying ring for formal groups. Thus, a one-dimensional formal group over a ring R is essentially equivalent to a complex genus, that is, a ring homomorphism $MU^* \to R$. One important example of such a genus is the Todd genus, a map $MU^* \to K^*$. The Todd genus occurs in the Hirzebruch–Riemann–Roch theorem, which calculates the index of the Dolbeault operator in terms of the Chern character. It also determines the K-theory of a finite space X from its complex cobordism groups, via the Conner–Floyd theorem: $K^*(X) \cong MU^*(X) \otimes_{MU^*} K^*$.

Elliptic curves form a natural source of formal groups, and hence complex genera. An example of this is Euler's formal group law over $\mathbb{Z}[\frac{1}{2}, \delta, \epsilon]$ associated to Jacobi's quartic elliptic curve; the corresponding elliptic cohomology theory is given on finite spaces by $X \mapsto MU^*(X) \otimes_{MU^*} \mathbb{Z}[\frac{1}{2}, \delta, \epsilon]$. Witten defined a genus $MSpin \to \mathbb{Z}[[q]]$ (not a complex genus, because not a map out of MU^*) which lands in the ring of modular forms, provided the characteristic class $\frac{p_1}{2}$ vanishes. He also gave an index theory interpretation of this genus, at a physical level of rigor, in terms of Dirac operators on loop spaces. It was later shown, by Ando–Hopkins–Rezk, that the Witten genus can be lifted to a map of ring spectra $MString \to tmf$. The theory of topological modular forms can therefore be seen as a solution to the problem of finding a kind of elliptic cohomology that is connected to the Witten genus in the same way that the Todd genus is to K-theory.

Chapter 2: Elliptic curves and modular forms. An elliptic curve is a non-singular curve in the projective plane defined by a Weierstrass equation:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Elliptic curves can also be presented abstractly, as pointed genus one curves. They are equipped with a group structure, where one declares the sum of three points to be zero if they are collinear in $\mathbb{P}^2$. The bundle of Kähler differentials on an elliptic curve, denoted ω, has a one-dimensional space of global sections.

When working over a field, one-dimensional group varieties can be classified into additive groups, multiplicative groups, and elliptic curves. However, when working over an arbitrary ring, the object defined by a Weierstrass equation will typically be a combination of those three cases. The general fibers will be elliptic curves, some fibers will be nodal (multiplicative groups), and some cuspidal (additive groups).

By a 'Weierstrass curve' we mean a curve defined by a Weierstrass equation—there is no smoothness requirement. An *integral modular form* can then be defined, abstractly, to be a law that associates to every (family of) Weierstrass curves a section of $\omega^{\otimes n}$, in a way compatible with base change. Integral modular forms form a graded ring, graded by the power of ω. Here is a concrete presentation of that ring:

$$\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta).$$

In the context of modular forms, the degree is usually called the *weight*: the generators c_4, c_6, and Δ have weight 4, 6, and 12, respectively. As we will see, those weights correspond to the degrees 8, 12, and 24 in the homotopy groups of tmf.

Chapter 3: The moduli stack of elliptic curves. We next describe the geometry of the moduli stack of elliptic curves over fields of prime characteristic, and over the integers. At large primes, the stack $\mathcal{M}_{ell}$ looks rather like it does over $\mathbb{C}$: