Tracts in Mathematics 7

Hans Triebel

Function Spaces and Wavelets on Domains

European Mathematical Society

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Preface

This book may be considered as a continuation of the monographs [T83], [T92], [T06]. Now we are mainly interested in spaces on domains in $\mathbb{R}^n$, related wavelet bases and wavelet frames, and extension problems. But first we deal in Chapter 1 with the usual spaces on $\mathbb{R}^n$, periodic spaces on $\mathbb{R}^n$ and on the n-torus $\mathbb{T}^n$ and their wavelet expansions under natural restrictions for the parameters involved. Spaces on arbitrary domains are the subject of Chapter 2. The heart of the exposition are the Chapters 3, 4, where we develop a theory of function spaces on so-called thick domains, including wavelet expansions and extensions to corresponding spaces on $\mathbb{R}^n$. This will be complemented in Chapter 5 by spaces on smooth manifolds and smooth domains. Finally we add in Chapter 6 a discussion about desirable properties of wavelet expansions in function spaces introducing the notation of Riesz wavelet bases and frames. This chapter deals also with some related topics, in particular with spaces on cellular domains.

Although we rely mainly on [T06] we repeat basic notation and a few classical assertions in order to make this text to some extent independently understandable and usable. More precisely, we have two types of readers in mind:

researchers in the theory of function spaces who are interested in wavelets as new effective building blocks, and

scientists who wish to use wavelet bases in *classical function spaces* in diverse applications.

Here is a ***guide*** to where one finds basic definitions and key assertions adapted to the second type of readers:

1. Classical Sobolev spaces $W_p^k(\mathbb{R}^n)$, Sobolev spaces $H_p^s(\mathbb{R}^n)$, classical Besov spaces $B_{pq}^s(\mathbb{R}^n)$ and Hölder–Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n)$ on the Euclidean n-spaces $\mathbb{R}^n$: Definition 1.1, Remark 1.2, p. 2.

2. Wavelets in $\mathbb{R}^n$: Section 1.2.1, p. 13.

3. Wavelet bases for spaces on $\mathbb{R}^n$: Theorem 1.20, p. 15.

4. Spaces on the n-torus $\mathbb{T}^n$: Definition 1.27, Remark 1.28, p. 21.

5. Wavelet bases for spaces on $\mathbb{T}^n$: Theorem 1.37, p. 26.

6. Spaces on arbitrary domains Ω in $\mathbb{R}^n$: Definitions 2.1, 5.17, Remark 2.2, pp. 28, 29, 147.

7. u-wavelet systems in domains Ω in $\mathbb{R}^n$: Definitions 2.4, 6.3, pp. 32, 179.

8. u-Riesz bases and u-Riesz frames: Definition 6.5, Section 6.2.2, pp. 180, 202.

9. Wavelet bases in $L_2(\Omega)$ and $L_p(\Omega)$ in arbitrary domains Ω in $\mathbb{R}^n$: Theorems 2.33, 2.36, 2.44, pp. 49, 53, 59.

10. Classes of domains Ω in $\mathbb{R}^n$ and their relations: Definitions 3.1, 3.4, 5.40, 6.9, Proposition 3.8, pp. 70, 72, 75, 168, 182.

11. Wavelet bases in E-thick domains (covering bounded Lipschitz domains) Ω in $\mathbb{R}^n$: Theorems 3.13, 3.23, Corollary 3.25, pp. 80, 89, 91.

12. Spaces, frames and bases on manifolds: Definitions 5.1, 5.5, 5.40, Theorems 5.9, 5.37, pp. 133, 135, 136, 164, 168.

13. Frames and bases on domains: Definition 5.25, Theorems 5.27, 5.35, 5.38, 5.51, 6.7, 6.30, 6.32, 6.33, pp. 152, 153, 162, 166, 175, 181, 196, 197, 198.

Formulas are numbered within chapters. Furthermore in each chapter all definitions, theorems, propositions, corollaries and remarks are jointly and consecutively numbered. Chapter n is divided in sections $n.k$ and subsections $n.k.l$. But when quoted we refer simply to Section $n.k$ or Section $n.k.l$ instead of Section $n.k$ or Subsection $n.k.l$. If there is no danger of confusion (which is mostly the case) we write A^s_{pq}, B^s_{pq}, F^s_{pq}, $\ldots$, a^s_{pq} ...(spaces) instead of $A^s_{p,q}$, $B^s_{p,q}$, $F^s_{p,q}$, $\ldots$, $a^s_{p,q}$ .... Similarly for a_{jm}, λ_{jm}, Q_{jm} (functions, numbers, cubes) instead of $a_{j,m}$, $\lambda_{j,m}$, $Q_{j,m}$ etc. References are ordered by names, not by labels, which roughly coincides, but may occasionally cause minor deviations. The numbers behind ► in the Bibliography mark the page(s) where the corresponding entry is quoted (with the exception of [T78]–[T06]).

It is a pleasure to acknowledge the great help I have received from David Edmunds (Brighton) who looked through the manuscript and offered many comments.

Jena, Summer 2008 Hans Triebel

Contents

Chapter 1
Spaces on $\mathbb{R}^n$ and $\mathbb{T}^n$

1.1 Definitions, atoms, and local means

1.1.1 Definitions

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the usual Schwartz space and $S'(\mathbb{R}^n)$ be the space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$, is the standard quasi-Banach space with respect to the Lebesgue measure in $\mathbb{R}^n$, quasi-normed by

$$\|f \,|L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \right)^{1/p} \tag{1.1}$$

with the natural modification if $p = \infty$. As usual, $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$ where $n \in \mathbb{N}$, denotes the lattice of all points $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Let $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, be the set of all multi-indices,

$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad \text{with } \alpha_j \in \mathbb{N}_0 \text{ and } |\alpha| = \textstyle\sum_{j=1}^n \alpha_j. \tag{1.2}$$

If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ then we put

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \quad \text{(monomials)}. \tag{1.3}$$

If $\varphi \in S(\mathbb{R}^n)$ then

$$\hat{\varphi}(\xi) = (F\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\, \varphi(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{1.4}$$

denotes the Fourier transform of φ. As usual, $F^{-1}\varphi$ and $\varphi^\vee$ stand for the inverse Fourier transform, given by the right-hand side of (1.4) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both F and F^{-1} are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi_0(y) = 0 \text{ if } |y| \ge 3/2, \tag{1.5}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n,\ k \in \mathbb{N}. \tag{1.6}$$

Since

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for } x \in \mathbb{R}^n, \tag{1.7}$$

the φ_j form a dyadic resolution of unity. The entire analytic functions $(\varphi_j \hat{f})^\vee(x)$ make sense pointwise in $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$.

Definition 1.1. Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be the above dyadic resolution of unity.

(i) Let

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}. \tag{1.8}$$

Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^\infty 2^{jsq} \|(\varphi_j \hat{f})^\vee \,|L_p(\mathbb{R}^n)\|^q \Big)^{1/q} < \infty \tag{1.9}$$

(with the usual modification if $q = \infty$).

(ii) Let

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}. \tag{1.10}$$

Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\|_\varphi = \Big\| \Big(\sum_{j=0}^\infty 2^{jsq} \,|(\varphi_j \hat{f})^\vee(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty \tag{1.11}$$

(with the usual modifications if $q = \infty$).

Remark 1.2. The theory of these spaces may be found in [T83], [T92], [T06]. In particular these spaces are independent of admitted resolutions of unity φ according to (1.5)–(1.7) (equivalent quasi-norms). This justifies our omission of the subscript φ in (1.9), (1.11) in the sequel. We remind the reader of a few special cases and properties referring for details to the above books, especially to [T06], Section 1.2.

(i) Let $1 < p < \infty$. Then

$$L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n) \tag{1.12}$$

is a well-known *Paley–Littlewood theorem*.

(ii) Let $1 < p < \infty$ and $k \in \mathbb{N}_0$. Then

$$W^k_p(\mathbb{R}^n) = F^k_{p,2}(\mathbb{R}^n) \tag{1.13}$$

are the *classical Sobolev spaces* usually equivalently normed by

$$\|f \,|W^k_p(\mathbb{R}^n)\| = \Big(\sum_{|\alpha| \le k} \|D^\alpha f \,|L_p(\mathbb{R}^n)\|^p \Big)^{1/p}. \tag{1.14}$$

This generalises (1.12).

(iii) Let $\sigma \in \mathbb{R}$. Then

$$I_\sigma : f \mapsto \big(\langle \xi \rangle^\sigma \hat{f} \,\big)^\vee \tag{1.15}$$

with $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, is an one-to-one map of $S(\mathbb{R}^n)$ onto itself and of $S'(\mathbb{R}^n)$ onto itself. Then I_σ is a lift for the spaces $A^s_{pq}(\mathbb{R}^n)$ with $A = B$ or $A = F$ and $s \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ for the F-scale), $0 < q \le \infty$:

$$I_\sigma A^s_{pq}(\mathbb{R}^n) = A^{s-\sigma}_{pq}(\mathbb{R}^n) \tag{1.16}$$

(equivalent quasi-norms). With

$$H^s_p(\mathbb{R}^n) = I_{-s} L_p(\mathbb{R}^n), \quad s \in \mathbb{R},\ 1 < p < \infty, \tag{1.17}$$

one has

$$H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n), \quad s \in \mathbb{R},\ 1 < p < \infty, \tag{1.18}$$

and

$$H^k_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n), \quad 1 < p < \infty,\ k \in \mathbb{N}_0. \tag{1.19}$$

Nowadays one calls the $H^s_p(\mathbb{R}^n)$ *Sobolev spaces* (sometimes fractional Sobolev spaces or Bessel-potential spaces) with the classical Sobolev spaces $W^k_p(\mathbb{R}^n)$ as a special case.

(iv) We denote

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n), \quad s \in \mathbb{R}, \tag{1.20}$$

as *Hölder–Zygmund spaces*. Let

$$(\Delta^1_h f)(x) = f(x+h) - f(x), \quad (\Delta^{l+1}_h f)(x) = \Delta^1_h (\Delta^l_h f)(x), \tag{1.21}$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $l \in \mathbb{N}$, be the iterated differences in $\mathbb{R}^n$. Let $0 < s < m \in \mathbb{N}$. Then

$$\| f \,|\mathcal{C}^s(\mathbb{R}^n)\|_m = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup |h|^{-s} |\Delta^m_h f(x)|, \tag{1.22}$$

where the second supremum is taken over all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$ with $0 < |h| \le 1$, are equivalent norms in $\mathcal{C}^s(\mathbb{R}^n)$.

(v) This assertion can be generalised as follows. Once more let $0 < s < m \in \mathbb{N}$ and $1 \le p, q \le \infty$. Then

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\|_m = \| f \,|L_p(\mathbb{R}^n)\| + \left(\int_{|h|\le 1} |h|^{-sq}\, \|\Delta^m_h f \,|L_p(\mathbb{R}^n)\|^q \frac{\mathrm{d}h}{|h|^n} \right)^{1/q} \tag{1.23}$$

and

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\|^*_m = \| f \,|L_p(\mathbb{R}^n)\| + \left(\int_0^1 t^{-sq} \sup_{|h|\le t} \|\Delta^m_h f \,|L_p(\mathbb{R}^n)\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} \tag{1.24}$$

(with the usual modification if $q = \infty$) are equivalent norms in $B^s_{pq}(\mathbb{R}^n)$. These are the *classical Besov spaces*. We refer to [T92], Chapter 1, and [T06], Chapter 1, where one finds the history of these spaces, further special cases and classical assertions. In addition, (1.23), (1.24) remain to be equivalent quasi-norms in

$$B^s_{pq}(\mathbb{R}^n) \quad \text{with } 0 < p, q \le \infty \text{ and } s > n\left(\tfrac{1}{p} - 1\right)_+. \tag{1.25}$$

1.1.2 Atoms

We give a detailed description of atomic representations of the spaces introduced in Definition 1.1 for two reasons. First we need these assertions later on. Secondly, atoms and local means, which are the subject of Section 1.1.3, are dual to each other where the natural smoothness assumptions and the cancellation conditions change their roles.

Let Q_{jm} be cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-j}m$ with side-length 2^{-j+1} where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side-length r times of the side-length of Q. Let χ_{jm} be the characteristic function of Q_{jm}.

Definition 1.3. Let $0 < p \le \infty$, $0 < q \le \infty$. Then b_{pq} is the collection of all sequences

$$\lambda = \{\lambda_{jm} \in \mathbb{C} : \, j \in \mathbb{N}_0, \, m \in \mathbb{Z}^n\} \tag{1.26}$$

such that

$$\|\lambda \,|b_{pq}\| = \Big(\sum_{j=0}^{\infty} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^p \Big)^{q/p} \Big)^{1/q} < \infty, \tag{1.27}$$

and f_{pq} is the collection of all sequences λ according to (1.26) such that

$$\|\lambda \,| f_{pq}\| = \Big\| \Big(\sum_{j,m} 2^{jnq/p} \, |\lambda_{jm} \, \chi_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n)\Big\| < \infty \tag{1.28}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

Remark 1.4. Note that the factor $2^{jnq/p}$ in (1.28) does not appear if one relies on the p-normalised characteristic function $\chi_{jm}^{(p)}(x) = 2^{n(j-1)/p} \chi_{jm}(x)$. This is the usual way to say what is meant by f_{pq}. But for what follows the above version seems to be more appropriate. Of course $b_{pp} = f_{pp}$.

Definition 1.5. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$ and $d \ge 1$. Then the L_∞-functions $a_{jm} : \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0, m \in \mathbb{Z}^n$, are called (s, p)-atoms if

$$\operatorname{supp} a_{jm} \subset d\, Q_{jm}, \quad j \in \mathbb{N}_0, \, m \in \mathbb{Z}^n; \tag{1.29}$$

there exist all (classical) derivatives $D^\alpha a_{jm}$ with $|\alpha| \le K$ such that

$$|D^\alpha a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})+j|\alpha|}, \quad |\alpha| \le K, \, j \in \mathbb{N}_0, \, m \in \mathbb{Z}^n, \tag{1.30}$$

and

$$\int_{\mathbb{R}^n} x^\beta \, a_{jm}(x) \, \mathrm{d}x = 0, \quad |\beta| < L, \, j \in \mathbb{N}, \, m \in \mathbb{Z}^n. \tag{1.31}$$

Remark 1.6. No cancellation (1.31) for $a_{0,m}$ is required. Furthermore, if $L = 0$ then (1.31) is empty (no condition). If $K = 0$ then (1.30) means $a_{jm} \in L_\infty$ and $|a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})}$. Of course, the conditions for the above atoms depend not only on s and p, but also on the given numbers K, L, d. But this will only be indicated when extra clarity is required. Otherwise we speak about (s, p)-atoms instead of $(s, p)_{K,L,d}$-atoms. Let as usual

$$\sigma_p = n\Big(\frac{1}{p} - 1\Big)_+ \quad \text{and} \quad \sigma_{pq} = n\Big(\frac{1}{\min(p,q)} - 1\Big)_+ \tag{1.32}$$

where $b_+ = \max(b, 0)$ if $b \in \mathbb{R}$.

Theorem 1.7. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, $d \in \mathbb{R}$ *with*

$$K > s, \quad L > \sigma_p - s, \tag{1.33}$$

and $d \ge 1$ *be fixed. Then* $f \in S'(\mathbb{R}^n)$ *belongs to* $B^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\, a_{jm} \tag{1.34}$$

where the a_{jm} *are* (s, p)*-atoms (more precisely* $(s, p)_{K,L,d}$*-atoms) according to Definition* 1.5 *and* $\lambda \in b_{pq}$. *Furthermore*

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|b_{pq}\| \tag{1.35}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.34) *(for fixed* K, L, d*).*

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, $d \in \mathbb{R}$ *with*

$$K > s, \quad L > \sigma_{pq} - s, \tag{1.36}$$

and $d \ge 1$ *be fixed. Then* $f \in S'(\mathbb{R}^n)$ *belongs to* $F^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented by* (1.34) *where* a_{jm} *are* (s, p)*-atoms (more precisely* $(s, p)_{K,L,d}$*-atoms) according to Definition* 1.5 *and* $\lambda \in f_{pq}$. *Furthermore,*

$$\| f \,|F^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|f_{pq}\| \tag{1.37}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.34) *(for fixed* K, L, d*).*

Remark 1.8. Recall that dQ_{jm} are cubes centred at $2^{-j}m$ with side-length $d\, 2^{-j+1}$ where $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. For fixed d with $d \ge 1$ and $j \in \mathbb{N}_0$ there is some overlap of the cubes dQ_{jm} where $m \in \mathbb{Z}^n$. This makes clear that Theorem 1.7 based on Definition 1.5 is reasonable. Otherwise the above formulations coincide essentially with [T06], Section 1.5.1. There one finds also some technical comments how the convergence in (1.34) must be understood. Atoms of the above type go back essentially to [FrJ85], [FrJ90]. But more details about the complex history of atoms may be found in [T92], Section 1.9.

1.1.3 Local means

We wish to derive estimates for local means which are dual to atomic representations according to Theorem 1.7 as far as smoothness assumptions and cancellation properties are concerned. First we collect some definitions. Let Q_{jm} be the same cubes as in the previous Section 1.1.2.

Definition 1.9. Let $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ and $C > 0$. Then the L_∞-functions $k_{jm}: \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called kernels (of local means) if

$$\operatorname{supp} k_{jm} \subset C Q_{jm}, \quad j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n; \tag{1.38}$$

there exist all (classical) derivatives $D^\alpha k_{jm}$ with $|\alpha| \le A$ such that

$$|D^\alpha k_{jm}(x)| \le 2^{jn + j|\alpha|}, \quad |\alpha| \le A, \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{1.39}$$

and

$$\int_{\mathbb{R}^n} x^\beta \, k_{jm}(x) \, \mathrm{d}x = 0, \quad |\beta| < B, \ j \in \mathbb{N}, \ m \in \mathbb{Z}^n. \tag{1.40}$$

Remark 1.10. No cancellation (1.40) for $k_{0,m}$ is required. Furthermore if $B = 0$ then (1.40) is empty (no condition). If $A = 0$ then (1.39) means $k_{jm} \in L_\infty$ and $|k_{jm}(x)| \le 2^{jn}$. Compared with Definition 1.5 for atoms we have different normalisations in (1.30) and (1.39) (also due to the history of atoms). We adapt the sequence spaces introduced in Definition 1.3 in connection with atoms to the above kernels.

Definition 1.11. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then $\bar{b}^s_{pq}$ is the collection of all sequences λ according to (1.26) such that

$$\|\lambda \,|\bar{b}^s_{pq}\| = \Big(\sum_{j=0}^\infty 2^{j(s-\frac{n}{p})q} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^p \Big)^{q/p} \Big)^{1/q} < \infty, \tag{1.41}$$

and $\bar{f}^s_{pq}$ is the collection of all sequences λ according to (1.26) such that

$$\|\lambda \,|\bar{f}^s_{pq}\| = \Big\| \Big(\sum_{j,m} 2^{jsq} \, |\lambda_{jm} \, \chi_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty \tag{1.42}$$

with the usual modification if $p = \infty$ and/or $q = \infty$.

Remark 1.12. The notation b^s_{pq} and f^s_{pq} (without bar) will be reserved for a slight modification of the above sequence spaces in connection with wavelet representations. One has $\bar{b}^s_{pp} = \bar{f}^s_{pp}$.

Definition 1.13. Let $f \in B^s_{pq}(\mathbb{R}^n)$ where $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Let k_{jm} be the kernels according to Definition 1.9 with $A > \sigma_p - s$ where σ_p is given by (1.32) and $B \in \mathbb{N}_0$. Then

$$k_{jm}(f) = (f, k_{jm}) = \int_{\mathbb{R}^n} k_{jm}(y) \, f(y) \, \mathrm{d}y, \quad j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{1.43}$$

are local means, considered as a dual pairing within $(S(\mathbb{R}^n), S'(\mathbb{R}^n))$. Furthermore,

$$k(f) = \{k_{jm}(f) : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}. \tag{1.44}$$

Remark 1.14. We justify the dual pairing (1.43). According to [T83], Theorems 2.11.2, 2.11.3, one has for the dual spaces of $B^s_{pp}(\mathbb{R}^n)$ that

$$B^s_{pp}(\mathbb{R}^n)' = B^{-s+\sigma_p}_{p'p'}(\mathbb{R}^n), \quad x \in \mathbb{R},\ 0 < p < \infty, \tag{1.45}$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{if } 1 \le p \le \infty \text{ and } p' = \infty \text{ if } 0 < p < 1. \tag{1.46}$$

Since $k_{jm} \in C^A(\mathbb{R}^n)$ ($L_\infty(\mathbb{R}^n)$ if $A = 0$) has compact support one obtains that $k_{jm} \in B^{A-\varepsilon}_{uv}(\mathbb{R}^n)$ for any $\varepsilon > 0$ and $0 < u, v \le \infty$. By

$$B^s_{pq}(\mathbb{R}^n) \subset B^{s-\varepsilon}_{pp}(\mathbb{R}^n) \quad \text{and} \quad B^s_{\infty q}(\mathbb{R}^n) \subset B^{s,\mathrm{loc}}_{pq}(\mathbb{R}^n) \tag{1.47}$$

locally for any $s \in \mathbb{R}$, $\varepsilon > 0$, $0 < p < \infty$ and $0 < q \le \infty$ one has by (1.45) and $A > \sigma_p - s$ that (1.43) makes always sense as a dual pairing. This applies also to $f \in F^s_{pq}(\mathbb{R}^n)$ since

$$F^s_{pq}(\mathbb{R}^n) \subset B^s_{p,\max(p,q)}(\mathbb{R}^n). \tag{1.48}$$

But (1.43) can also be justified for $f \in B^s_{pq}(\mathbb{R}^n)$ and $f \in F^s_{pq}(\mathbb{R}^n)$ as in [T06], Section 5.1.7, by direct arguments.

In Section 1.2 we characterise the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ in terms of wavelets. Since wavelets are special atoms one has by Theorem 1.7 the desired estimates from above. But wavelets can also be considered as kernels k_{jm} of corresponding local means. This gives finally the needed estimates from below which will be reduced to the following theorem which might be considered as the main assertion of Section 1.1.

Theorem 1.15. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* k_{jm} *be kernels according to Definition* 1.9 *where* $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ *with*

$$A > \sigma_p - s, \quad B > s, \tag{1.49}$$

and $C > 0$ *are fixed. Let* $k(f)$ *be as in* (1.43), (1.44). *Then for some* $c > 0$ *and all* $f \in B^s_{pq}(\mathbb{R}^n)$,

$$\|k(f)\,|\bar{b}^s_{pq}\| \le c\,\|f\,|B^s_{pq}(\mathbb{R}^n)\|. \tag{1.50}$$

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* k_{jm} *and* $k(f)$ *be the above kernels where* $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ *with*

$$A > \sigma_{pq} - s, \quad B > s, \tag{1.51}$$

and $C > 0$ *are fixed. Then for some* $c > 0$ *and all* $f \in F^s_{pq}(\mathbb{R}^n)$,

$$\|k(f)\,|\bar{f}^s_{pq}\| \le c\,\|f\,|F^s_{pq}(\mathbb{R}^n)\|. \tag{1.52}$$