NONLINEAR
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SCIENCE

Sergey P. Kuznetsov

Hyperbolic Chaos

A Physicist's View

双曲混沌

一个物理学家的观点

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Preface

One of intensively developing research directions is exploration of complex dynamics and chaos in nonlinear systems. We regard an object as a *dynamical system*, if its state at any given time may be obtained from the initial state according to a certain rule defined for the given system. It is remarkable that such definition, although representing ideal of deterministic vision, does not exclude a possibility of *chaotic behavior* of the object, with dependence of the state on time looking like a random process. Chaos occurs in systems of different nature, e.g. relating to mechanics, hydrodynamics, electronics, laser physics and nonlinear optics, chemical kinetics, and biomedical disciplines. The main attribute of the dynamical chaos is *sensitivity to small perturbations of initial conditions*, which makes it impossible to predict the future states actually for times longer than some characteristic scale, which usually depends logarithmically on the inaccuracy of the initial conditions ("horizon of predictability").

For dissipative dynamical systems, chaos is associated with presence in the state space of a curious object called *the strange attractor*. At present, the collection of models with strange attractors is very rich, including artificial mathematical examples, models of physical, chemical, and biological systems.

A classic example of chaotic attractor is *Lorenz attractor*. It occurs in a set of three first-order differential equations modeling fluid convection or dynamics of a single-mode laser. The Lorenz model was a subject of extensive and careful studies in recent decades. Accurate mathematical foundation of chaotic nature of dynamics of the Lorenz model was found to be a subtle and delicate problem. It was announced as the 14-th Smale problem (from the list of difficult mathematical problems suggested by Steven Smale in 1998, as challenging mathematicians of the 21st century, similar to the list of Hilbert problems advanced for the 20th century). The solution was given by W. Tucker in 2002 by combination of technique of computer-assisted proof and careful analytic considerations.

Maybe, there was a lost opportunity for the nonlinear science to find out an alternative, a less painful way to discover physically meaningful and mathematically validated examples of chaotic behavior.

About 40 years ago, in mathematical studies a special type of chaotic attractors was introduced, namely, the *uniformly hyperbolic attractors*. They relate to systems belonging to the so-called *axiom A class* and are considered in the *hyperbolic theory* associated with the names of Anosov, Alekseev, Smale, Williams, Sinai, Plykin, Ruelle, Pesin, Newhouse and others. Chaotic nature of dynamics on the uniformly hyperbolic attractors is proved rigorously. They possess a property of *structural stability*, i.e., the phase space arrangement, character of dynamics, and its statistical characteristics are insensitive to variation of parameters and functions in the governing equations.

Originally, it was expected that the uniformly hyperbolic attractors will be relevant to many physical situations of occurrence of dynamical chaos. However, as time passed and many examples of chaotic systems of different nature had been suggested and studied, it became clear that these examples do not fit the narrow frames of the early hyperbolic theory. Therefore, people started to think of the uniformly hyperbolic attractors rather as of refined abstract image of chaos that has no relation to real systems. Efforts of mathematicians were redirected at developing generalizations applicable to broader classes of systems. For example, notions of singular-hyperbolic (or quasi-hyperbolic) attractors, non-uniformly hyperbolic attractors, partially hyperbolic attractors, and quasi-attractors were introduced, and certain progress in their exploration was achieved.

Abandoned for a long time and not clarified until recently is a question on a possibility of occurrence of the dynamical behavior associated with uniformly hyperbolic attractors in real world systems or, at least, in specially designed systems in physics and technology.

In textbooks on and reviews of nonlinear dynamics, the uniformly hyperbolic attractors are represented usually by artificial discrete time models based on geometric constructions, often explained qualitatively, in words, or by graphic images. Of course, for a physicist this is only a starting point for a research.

First of all, in addition to the geometric constructions it is desirable to have examples in a form of explicitly written equations, which allow application of computer methods for analyzing the dynamics and calculation of quantitative characteristics interesting to possible applications.

For some physical systems description in discrete time is very natural, and it would be worth examining possibilities of occurrence of the hyperbolic attractors in maps relating to such systems.

Next, it is important to turn to continuous time systems, which are of the first place importance, e.g., in physics and technology.

It is desirable to have a clear vision of how to implement the dynamics on uniformly hyperbolic attractors using combinations of structural elements known in the context of the theory of oscillations and in applications (oscillators, coupled systems, and feedback loops).

Finally, the proposed models should be implemented as real operating devices, for example, in electronics, mechanics, nonlinear optics, and technical applications of such devices should be indicated with explanation of advantages over alternative possible solutions.

In the theory of oscillations, since classic works of Andronov and his school, rough or structurally stable systems are regarded as those subjected to priority research, and as the most important for practice. It seems natural that the same should relate to systems with structurally stable uniformly hyperbolic attractors. The lack of physical examples in this regard looks as an evident dissonance. From a methodological point of view, the situation is similar to that in the early 20th century concerning the limit cycles, before their role as mathematical images for self-oscillations was established. In a similar way, the uniformly hyperbolic chaotic attractors should find their place as images for phenomena in real systems. It will help to link the abstract hyperbolic theory developed by mathematicians with description of real systems and attribute this theory to physical content.

This book is devoted to review the modern state of the outlined research program. The material is presented in four parts.

The introductory Part I consists of two chapters. Chapter 1 is devoted to the necessary basic concepts, including the notion of dynamical system, attractor, Poincaré map, and hyperbolicity. We introduce and discuss classic examples of uniformly hyperbolic attractors: Smale-Williams solenoid, DA-attractor of Smale, and attractors of Plykin type. An overview is given to the content of the hyperbolic theory (the cone criterion, structural stability, Markov partitions and symbolic dynamics, measures of Sinai-Ruelle-Bowen, etc.). Chapter 2 is a review of the literature indicating various possible situations of occurrence of uniformly hyperbolic attractors in dynamical systems.

Part II is the basic one. Here, we introduce a number of examples of systems, which allow physical implementation, and possess uniformly hyperbolic attractors with one-dimensional unstable manifolds (one positive Lyapunov exponent). In Chap. 3 we consider models, which admit a natural stroboscopic discrete time description. These are systems operating under periodic pulse kicks and systems whose dynamics is composed of a periodic sequence of stages, each of which is governed by a particular form of the differential equations. Chapter 4 discusses models designed on a base of two self-oscillators which are excited alternately because of forced external parameter modulation and transmit the excitation to each other in such way that its phase transforms in accordance with expanding circle map. Chapter 5 is devoted to autonomous systems operating according to the same principle. In Chap. 6 we consider schemes of parametric generators of chaos with hyperbolic attractors. Chapter 7 is devoted to methods of computer verification of the hyperbolic nature of attractors illustrated with the examples suggested in the previous chapters. Particularly, we consider technique of visualization of mutual location of stable and unstable manifolds, statistics of angles of intersections between them, visualization of the natural invariant measure distributions on the attractors, and verification of the cone criterion.

Part III is devoted to model systems, for which mathematical justification of the hyperbolic nature of the attractors is more problematic. Hypothetically, however, the hyperbolicity apparently takes place. This assertion is based on the fact that from a physical point of view, those models are built following the same principles as the systems considered in Chaps. 4 and 5, for which the hyperbolicity is justified at the level of computations. In Chap. 8 we consider non-autonomous systems of four alternately excited oscillators. Among them, there is a model, in which the transformation of phases during a period of parameter modulation follows the Anosov map on the torus, and a model associated with a map on the torus with hyperchaotic dynamics (more than one positive Lyapunov exponent). Also a system is considered constructed of two coupled elements, with each possessing a hyperbolic attractor. For this case, some details are studied for the transition corresponding to complete synchronization of chaos. In Chap. 9, an autonomous system is considered functioning due to dynamics nearby a heteroclinic cycle in the amplitude equations. On this basis, we implement an attractor of Smale-Williams type in the Poincaré map, an attractor with dynamics of cyclic (phase) variables governed by the Anosov map, and a hyperchaotic attractor. Chapter 10 is devoted to systems with delayed feedback, in which a chaotic map for the phases of successively generated oscillation trains is arranged due to the transfer of excitation from the previous stages of activity of a single oscillatory element (self-oscillator) to the next stages. In Chap. 11 we consider a high-dimensional system built on two ensembles of self-oscillators with global coupling within each of them. Due to the turning the coupling on and off alternately in two ensembles, they undergo periodic Kuramoto transitions from synchronization to desynchronization and back; the interaction between the ensembles is organized in such way that the phase of the oscillatory excitation for the mean field transferred to each other, follows the chaotic map.

Part IV is devoted to examples of hyperbolic (or hypothetically hyperbolic) attractors in electronic experiments available to date. In Chap. 12 we consider experiments with a non-autonomous system of two alternately excited self-oscillators manifesting dynamics on the attractor of Smale-Williams type. In Chap. 13 two variants of non-autonomous time-delay systems with periodic parameter modulation are discussed. One possesses presumably an attractor of Smale-Williams type, embedded in an infinite-dimensional state space of the respective stroboscopic Poincaré map, and the other corresponds to a partially hyperbolic attractor.

The Appendix address some matters, which are essential to the whole presentation and illustrations, but dropping out of the basic structure of the book. We consider algorithm for computing Lyapunov exponents based on the Gram-Schmidt orthogonalization, and derivation of the Hénon and Ikeda maps from physical considerations for systems with impulse kicks. Further, the construction of the Smale horseshoe is presented, which gives a nontrivial example of a non-attractive complicated invariant set. The Kaplan-Yorke formula is derived and explained connecting approximately the Lyapunov exponents and dimensions of chaotic attractors. Formal definition of the model of Hunt is reproduced, which implements suspension of attractor of Plykin type. A simple model of hyperbolic dynamics on a compact surface of negative curvature is considered. The final Appendix is devoted to the shadowing property of hyperbolic chaotic attractors, which is illustrated in computations for a model from Chap. 4 with added noise.

The book may be useful to physicists and engineers interested in the practical application of the theory of deterministic chaos, particularly in obtaining robust chaos insensitive to parameters and characteristics of components, fluctuations, interferences, etc. This may relate to various disciplines——mechanics, hydrodynamics, electronics, laser physics, and nonlinear optics.

Contents of the book can probably help to think about possibilities of occurrence of structurally stable chaos in systems of different physical nature. Intriguing area for contemplations may be the significance of such chaos in neurodynamics.

On the other hand, the book can be useful for mathematicians interested in concrete applications of the hyperbolic theory. For them, it may be of interest to see how the mathematical theories are refracted from the perspective of the applied disciplines.

The author tried to present the material in a style available to graduate and post graduate students of non-mathematical specialties and to make it, as far as possible, self-consistent to allow a study without recourse to other sources. I tried to avoid formal definitions and formulations with abusing mathematical symbolism, replacing them with rather intuitive qualitative arguments. Perhaps, a part of mathematically oriented readers will find it insufficient; they are referred to rich literature on mathematical theory of dynamical systems. Also, I must warn that the consideration of general content of nonlinear dynamics is restricted here by the minimum needed for understanding the substantive material of the book. So, it can not be regarded as a substitute for a systematic study of integral general courses of nonlinear dynamics, to which the interested reader is referred for this purpose.

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Sergey P. Kuznetsov

Saratov, February 2011

Contents

Part I Basic Notions and Review

Part I
Basic Notions and Review

Chapter 1
Dynamical Systems and Hyperbolicity

Abstract In this chapter we review basic notions of the theory of dynamical systems essential for understanding subsequent chapters of the book. Particularly, a definition of the dynamical system is discussed and some important classes like continuous-time and discrete-time systems, conservative and dissipative systems, autonomous and non-autonomous systems are introduced. Interpretation of dynamics as evolution of a cloud of representative points in the state space is considered and implemented for explanation of chaos and, particularly, for the uniformly hyperbolic attractors, like Smale-Williams solenoid, DA attractor of Smale, and Plykin type attractors. Lyapunov exponents as a tool for quantitative approach to analyzing chaotic or regular dynamics are examined, and methodic of their computation is discussed. Main notions of the hyperbolic theory are considered, and its substational content is briefly reviewed.

1.1 Dynamical systems: basic notions

1.1.1 Systems with continuous and discrete time, and their mutual relation

A finite-dimensional dynamical system is an object, for which one can specify a state by a collection of a finite number N of real variables named a *state vector*, supposing that there exists a definite rule called the *evolution operator* that makes it possible to indicate precisely the state vector resulting from the initial one at any latter time instant (Birkhoff, 1927; Schuster and Just, 2005; Thompson and Stewart, 1986; Strogatz, 2001; Hasselblatt and Katok, 2003). A set of all possible states is a *phase space*, or *state space*; its dimension equals just the number of variables N needed to specify the state vector. In other words, this is a space with coordinate axes, each of which is associated with one dynamical variable from the set of N of them.

Both continuous-time and discrete-time systems are introduced and considered; in mathematical literature they are called *flows* and *cascades*, respectively.

Evolution of a state in time corresponds to motion of the representative point in the phase space along the *phase trajectory*, or *orbit*. A set in the phase space is called *invariant set* if all representative points from this set are transformed by the evolution operator again to the points belonging to the same set.

In order to describe continuous-time *autonomous* systems, one can use differential equations of the form

$$d\mathbf{x}/dt = \mathbf{F}(\mathbf{x}). \tag{1.1}$$

Here $\mathbf{x} = (x_1, x_2, \dots, x_N)$ is a state vector of dimension N, and $\mathbf{F} = (F_1(\mathbf{x}), F_2(\mathbf{x}), \dots, F_N(\mathbf{x}))$ is a vector function. In coordinate notation,

$$\begin{aligned} dx_1/dt &= F_1(x_1, x_2, \dots, x_N), \\ dx_2/dt &= F_2(x_1, x_2, \dots, x_N), \\ &\cdots\cdots\cdots\cdots \\ dx_N/dt &= F_N(x_1, x_2, \dots, x_N). \end{aligned} \tag{1.2}$$

By virtue of the theorem of existence and uniqueness of solutions for sets of differential equations (Arnold, 1978), with a given state at some instant t_0, one can determine states in the future $t > t_0$, as well as in the past $t < t_0$. In other words, the evolution of a state may be monitored both forward and backward in time.

If the right-hand side function $\mathbf{F}$ in the differential equation depends on time explicitly, the system is called *non-autonomous*. Physically, it corresponds to systems operating in the presence of external time-dependent forcing. To specify the state in this case, besides the vector $\mathbf{x}$ one needs to indicate the time instant it relates to. Therefore, we introduce the space of dimension $N+1$ with an additional coordinate axis t, which in this context is called the *extended phase space*. In this book, referring to non-autonomous systems, we will consider only the class of systems with the functions periodic in time: $\mathbf{F}(\mathbf{x}, t+T) = \mathbf{F}(\mathbf{x}, t)$.

Discrete time systems are described by evolution rules defined by *iterated maps*, which correspond to transformation of the states step by step,

$$\mathbf{x}_{n+1} = \mathbf{g}(\mathbf{x}_n). \tag{1.3}$$

Here $\mathbf{x}$ is a state vector, and $\mathbf{g}$ is a vector function specifying the evolution operator. In this case, a phase trajectory is a discrete sequence of points in the phase space. Evolution that takes place for k steps corresponds to k-fold iteration of the map; it is designated as

$$\mathbf{x}_{n+k} = \mathbf{g}^k(\mathbf{x}_n) \equiv \underbrace{\mathbf{g}(\mathbf{g}(\cdots \mathbf{g}}_{k\ \text{times}}(\mathbf{x}_n)\cdots)) \equiv \underbrace{\mathbf{g} \circ \mathbf{g} \circ \cdots \circ \mathbf{g}}_{k\ \text{times}}(\mathbf{x}_n). \tag{1.4}$$

Continuous time systems and discrete time systems are closely related. Procedure of passage from one to another is known as the *Poincaré section* construction. In phase space of a continuous time system one selects some fixed surface in such a way that the phase trajectories cross it again and again in the course of the time