Graduate Texts in Mathematics

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Classical Topology and Combinatorial Group Theory

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John Stillwell

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Second Edition

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John Stillwell
Department of Mathematics
Monash University
Clayton (Victoria 3168)
Australia

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Second Edition

Illustrated with 312 Figures by the Author

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To my mother
and father

Preface to the First Edition

In recent years, many students have been introduced to topology in high school mathematics. Having met the Möbius band, the seven bridges of Königsberg, Euler's polyhedron formula, and knots, the student is led to expect that these picturesque ideas will come to full flower in university topology courses. What a disappointment "undergraduate topology" proves to be! In most institutions it is either a service course for analysts, on abstract spaces, or else an introduction to homological algebra in which the only geometric activity is the completion of commutative diagrams. Pictures are kept to a minimum, and at the end the student still does not understand the simplest topological facts, such as the reason why knots exist.

In my opinion, a well-balanced introduction to topology should stress its intuitive geometric aspect, while admitting the legitimate interest that analysts and algebraists have in the subject. At any rate, this is the aim of the present book. In support of this view, I have followed the historical development where practicable, since it clearly shows the influence of geometric thought at all stages. This is *not* to claim that topology received its main impetus from geometric recreations like the seven bridges; rather, it resulted from the *visualization* of problems from other parts of mathematics—complex analysis (Riemann), mechanics (Poincaré), and group theory (Dehn). It is these connections to other parts of mathematics which make topology an important as well as a beautiful subject.

Another outcome of the historical approach is that one learns that classical (prior to 1914) ideas are still alive, and still being worked out. In fact, many simply stated problems in 2 and 3 dimensions remain unsolved. The development of topology in directions of greater generality, complexity, and abstractness in recent decades has tended to obscure this fact.

Attention is restricted to dimensions ≤ 3 in this book for the following reasons.

(1) The subject matter is close to concrete, physical experience.
(2) There is ample scope for analytic, geometric, and algebraic ideas.
(3) A variety of interesting problems can be constructively solved.
(4) Some equally interesting problems are still open.
(5) The combinatorial viewpoint is known to be completely general.

The significance of (5) is the following. Topology is ostensibly the study of arbitrary continuous functions. In reality, however, we can comprehend and manipulate only functions which relate finite "chunks" of space in a simple combinatorial manner, and topology originally developed on this basis. It turns out that for figures built from such chunks (simplexes) of dimension ≤ 3, the combinatorial relationships reflect all relationships which are topologically possible. Continuity is therefore a concept which can (and perhaps should) be eliminated, though of course some hard foundational work is required to achieve this.

I have not taken the purely combinatorial route in this book, since it would be difficult to improve on Reidemeister's classic *Einführung in die Kombinatorische Topologie* (1932), and in any case the relationship between the continuous and the discrete is extremely interesting. I have chosen the middle course of placing one combinatorial concept—the fundamental group—on a rigorous foundation, and using others such as the Euler characteristic only descriptively. Experts will note that this means abandoning most of homology theory, but this is easily justified by the saving of space and the relative uselessness of homology theory in dimensions ≤ 3. (Furthermore, textbooks on homology theory are already plentiful, compared with those on the fundamental group.)

Another reason for the emphasis on the fundamental group is that it is a two-way street between topology and algebra. Not only does group theory help to solve topological problems, but topology is of genuine help in group theory. This has to do with the fact that there is an underlying computational basis to both combinatorial topology and combinatorial group theory. The details are too intricate to be presented in this book, but the relevance of computation can be grasped by looking at topological problems from an algorithmic point of view. This was a key concern of early topologists and in recent times we have learned of the *nonexistence* of algorithms for certain topological problems, so it seems timely for a topology text to present what is known in this department.

The book has developed from a one-semester course given to fourth year students at Monash University, expanded to two-semester length. A purely combinatorial course in surface topology and group theory, similar to the one I originally gave, can be extracted from Chapters 1 and 2 and Sections 4.3, 5.2, 5.3, and 6.1. It would then be perfectly reasonable to spend a second semester deepening the foundations with Chapters 0 and 3 and going on to 3-manifolds in Chapters 6, 7, and 8. Certainly the reader is not obliged to master Chapter 0 before reading the rest of the book. Rather, it should be skimmed once and then referred to when needed later. Students who have had a conventional first course in topology may not need 0.1–0.3 at all.

The only prerequisites are some familiarity with elementary set theory, coordinate geometry and linear algebra, ε-δ arguments as in rigorous calculus, and the group concept.

The text has been divided into numbered sections which are small enough, it is hoped, to be easily digestible. This has also made it possible to dispense with some of the ceremony which usually surrounds definitions, theorems, and proofs. Definitions are signalled simply by italicizing the terms being defined, and they and proofs are not numbered, since the section number will serve to locate them and the section title indicates their content. Unless a result already has a name (for example, the Seifert-Van Kampen theorem) I have not given it one, but have just stated it and followed with the proof, which ends with the symbol □.

Because of the emphasis on historical development, there are frequent citations of both author and date, in the form: Poincaré 1904. Since either the author or the date may be operative in the sentence, the result is sometimes grammatically curious, but I hope the reader will excuse this in the interests of brevity. The frequency of citations is also the result of trying to give credit where credit is due, which I believe is just as appropriate in a textbook as in a research paper. Among the references which I would recommend as parallel or subsequent reading are Giblin 1977 (homology theory for surfaces), Moise 1977 (foundations for combinatorial 2- and 3-manifold theory), and Rolfsen 1976 (knot theory and 3-manifolds).

Exercises have been inserted in most sections, rather than being collected at the ends of chapters, in the hope that the reader will do an exercise more readily while his mind is still on the right track. If this is not sufficient prodding, some of the results from exercises are used in proofs.

The text has been improved by the remarks of my students and from suggestions by Wilhelm Magnus and Raymond Lickorish, who read parts of earlier drafts and pointed out errors. I hope that few errors remain, but any that do are certainly my fault. I am also indebted to Anne-Marie Vandenberg for outstanding typing and layout of the original manuscript.

October 1980 JOHN C. STILLWELL

Preface to the Second Edition

There have been several big developments in topology since the first edition of this book. Most of them are too difficult to include here, or else, well written up elsewhere, so I shall merely mention below what they are and where they may be found. The main new inclusion in this edition is a proof of the unsolvability of the word problem for groups, and some of its consequences. This is made possible by a new approach to the word problem discovered by Cohen and Aanderaa around 1980. Their approach makes it feasible to prove a series of unsolvability results we previously mentioned without proof, and thus to tie up several loose ends in the first edition. A new Chapter 9 has been added to incorporate these results. It is particularly pleasing to be able to give a proof of the unsolvability of the homeomorphism problem, which has not previously appeared in a textbook.

What are the other big developments? They would have to include the proof by Freedman in 1982 of the 4-dimensional Poincaré conjecture, and the related work of Donaldson on 4-manifolds. These difficult results may be found in Freedman and Quinn's *The Topology of 4-manifolds* (Princeton University Press, 1990) and Donaldson and Kronheimer's *The Geometry of Four-Manifolds* (Oxford University Press, 1990). With Freedman's proof, only the original (3-dimensional) Poincaré conjecture remains open. In fact, the main problems of 3-dimensional topology seem to be just as stubborn as they were in 1980. There is still no algorithm for deciding when 3-manifolds are homeomorphic, or even for recognizing the 3-sphere. However, there has been important progress in knot theory, most of which stems from the *Jones polynomial*, a new knot invariant found by Jones in 1983. For a sampling of this rapidly growing field, and its mysterious connections with physics, see Kauffman's *Knots and Physics* (World Scientific, 1991).

Recent developments in combinatorial group theory are a natural continuation of two themes in the present book—the tree structure behind free groups and the tessellation structure behind Dehn's algorithm. The main results on tree structure and its generalizations may be found in Dicks and Dunwoody's *Groups Acting on Graphs* (Cambridge University Press, 1989). Dehn's algorithm has been generalized to many other groups which act on tessellations with combinatorial properties like those discovered by Dehn in the hyperbolic plane (see *Group Theory from a Geometrical Viewpoint*, edited by Ghys, Haefliger and Verjovsky, World Scientific, 1991). Both these lines of research should be accessible to readers of the present book, though a little more preparation is advisable. I recommend Serre's *Trees* (Springer-Verlag, 1980) and Dehn's *Papers in Group Theory and Topology* (Springer-Verlag, 1987). My own *Geometry of Surfaces* (Springer-Verlag, 1992) may also serve as a source for hyperbolic geometry, and as a replacement for the very sketchy account of geometric methods given in 6.2 below.

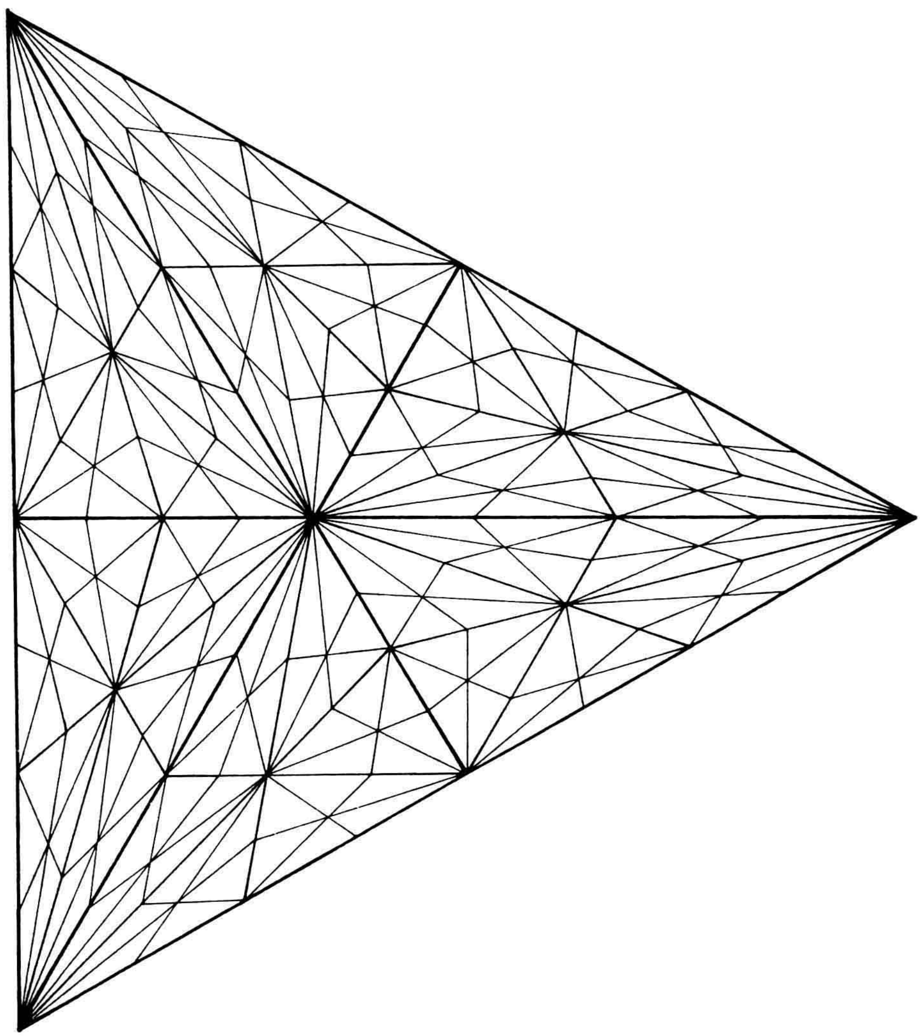
Finally, I should mention that this edition includes numerous corrections sent to me by readers. I am particularly grateful to Peter Landweber, who contributed the most thorough critique, as well as encouragement for a second edition.

Clayton, November 1992 JOHN C. STILLWELL

Contents

CHAPTER 0

Introduction and Foundations



0.1 The Fundamental Concepts and Problems of Topology

0.1.1 The Homeomorphism Problem

Topology is the branch of geometry which studies the properties of figures under arbitrary continuous transformations. Just as ordinary geometry considers two figures to be the same if each can be carried into the other by a rigid motion, topology considers two figures to be the same if each can be mapped onto the other by a one-to-one continuous function. Such figures are called topologically equivalent, or *homeomorphic*, and the problem of deciding whether two figures are homeomorphic is called the *homeomorphism problem.*

One may consider a geometric figure to be an arbitrary point set, and in fact the homeomorphism problem was first stated in this form, by Hurwitz 1897. However, this degree of generality makes the problem completely intractable, for reasons which belong more to set theory than geometry, namely the impossibility of describing or enumerating all point sets. To discuss the problem sensibly we abandon the elusive "arbitrary point set" and deal only with *finitely describable* figures, so that a solution to the homeomorphism problem can be regarded as an *algorithm* (0.4) which operates on descriptions and produces an answer to each homeomorphism question in a finite number of steps.

The most convenient building blocks for constructing figures are the simplest euclidean space elements in each dimension:

dimension 0: point
dimension 1: line segment
dimension 2: triangle
dimension 3: tetrahedron
⋮

We call the simplest space element in n-dimensional euclidean space R^n the *n-simplex* Δ^n. It is constructed by taking $n + 1$ points $P_1, \ldots, P_{n+1}$ in R^n which do not lie in the same $(n - 1)$-dimensional hyperplane, and forming their *convex hull*; that is, closing the set under the operation which fills in the line segment between any two points. In algebraic terms, we take $n + 1$ linearly independent vectors $\boldsymbol{OP}_1, \ldots, \boldsymbol{OP}_{n+1}$ (where $\boldsymbol{OP}_i$ denotes the vector from the origin O to P_i) and let Δ^n consist of the endpoints of the vectors

$$x_1\boldsymbol{OP}_1 + \cdots + x_{n+1}\boldsymbol{OP}_{n+1},$$

where $x_1 + \cdots + x_{n+1} = 1$ and $x_i \geq 0$. It is now an easy exercise (0.1.1.1 below) to show that any two n-simplexes are homeomorphic, so we are entitled to speak of *the* n-simplex Δ^n.

Each subset of $m + 1$ points from $\{P_1, \ldots, P_{n+1}\}$ similarly determines an *m-dimensional face* Δ^m of Δ^n. The union of the $(n - 1)$-dimensional faces is called the *boundary* of Δ^n, so all lower-dimensional faces lie in the boundary. We shall build figures, called *simplicial complexes*, by pasting together simplexes so that faces of a given dimension are either disjoint or coincide completely. This method of construction, which is due to Poincaré 1899, will be studied more thoroughly in 0.2. For the moment we wish to claim that all "natural" geometric figures are either simplicial complexes or homeomorphic to them, which is just as good for topological purposes.

This claim is supported by some figures which play a prominent role in this book—surfaces and knots. Surfaces may be constructed by pasting triangles together, so they are simplicial complexes of dimension 2. For example, the surface of a tetrahedron (which is homeomorphic to a sphere) is a simplicial complex of four triangles as shown in Figure 1. The torus surface (Figure 2) can be represented as a simplicial complex as shown in Figure 3. The representation is of course not unique, and from this one begins to see the *combinatorial* core of the homeomorphism problem, which remains after the point set difficulties have been set aside. Given a description of a surface as a list of triangles and their edges, how does one assess its *global* form? In particular, are the sphere and the torus topologically different? In fact we know how to solve this problem (by the classification theorem of 1.3, and 5.3.3), but not the corresponding 3-dimensional problem.

Much of the difficulty in dimension 3 is due to the existence of *knots*. We could define a knot to be any simple closed curve $\mathscr{K}$ in R^3, but any such

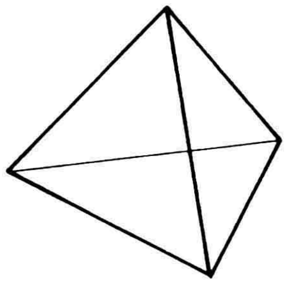


Figure 1

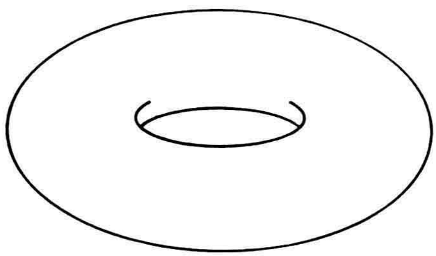


Figure 2

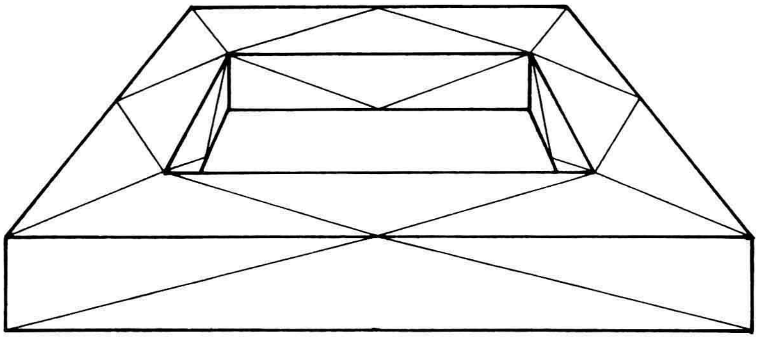


Figure 3

$\mathscr{K}$ is homeomorphic to a circle and its "knottedness" actually resides in the complement space $\mathbf{R}^3 - \mathscr{K}$. This space is not finitely describable in terms of simplexes, so we replace $\mathbf{R}^3$ by, say, a cube and drill a thin tube out of it following the "knotted part" of $\mathscr{K}$ (see Figure 4).

This figure can be divided into small tetrahedra and hence is a finite simplicial complex representing the knot. The homeomorphism problem for such figures is extraordinarily difficult; Riemann was perhaps the first to think about it seriously (see Weil 1979), and it has been solved only recently (see Hemion 1979, Waldhausen 1978). The solution extends to more general "knot spaces" obtained by drilling any number of tubes out of cubes, but not as yet to all the figures which result from pasting knot spaces together.

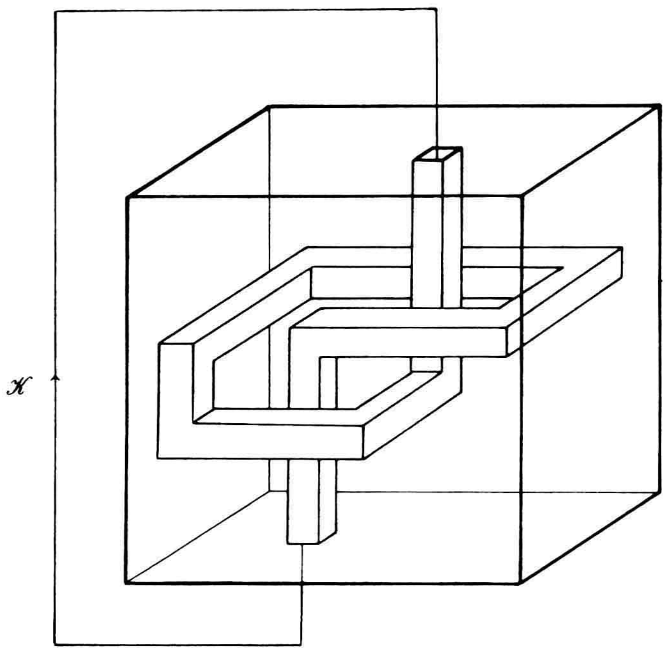


Figure 4