SECOND EDITION

Real Analysis

Serge Lang

REAL ANALYSIS

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Foreword

This book is meant as a text for a first year graduate course in analysis. Any standard course in undergraduate analysis will constitute sufficient preparation for its understanding, for instance my *Undergraduate Analysis*. I assume that the reader is acquainted with notions of uniform convergence and the like.

In a sense, the subject matter covers the same topics as elementary calculus, viz. linear algebra, differentiation and integration. This time, however, these subjects are treated in a manner suitable for the training of professionals, i.e. people who will use the tools in further investigations, be it in mathematics, or physics, or what have you.

In the first part, we begin with point set topology, essential for all analysis, and we cover the most important results.

I am selective here, since this part is regarded as a tool, especially Chapters 1 and 2. Many results are easy, and are less essential than those in the text. They have been given in exercises, which are designed to acquire facility in routine techniques and to give flexibility for those who want to cover some of them at greater length. The point set topology simply deals with the basic notions of continuity, open and closed sets, connectedness, compactness, and continuous functions. The chapter concerning continuous functions on compact sets properly emphasizes results which already mix analysis and uniform convergence with the language of point set topology.

The differential calculus is done because at best, most people will only be acquainted with it only in Euclidean space, and incompletely at that. More importantly, the calculus in Banach spaces has acquired considerable importance in the last two decades, because of many applications like Morse theory, the calculus of variations, and the Nash-Moser implicit mapping theorem, which lies even further in this direction since one has to deal with more general spaces than Banach spaces. These results pertain to the geometry of function spaces. Cf. the exercises of Chapter 6 for simpler applications.

Next, we cover some functional analysis. The purpose here is twofold. We place the linear algebra in an infinite dimensional setting where continuity assumptions are made on the linear maps, and we show how one can "linearize" a problem by taking derivatives, again in a setting where the theory can be ultimately applied to function spaces. Chapters 4, 7, 9, and 10, which include two major spectral theorems of analysis, show how we can extend to the infinite dimensional case certain results of finite dimensional linear algebra. The compact and Fredholm operators lately have been receiving renewed attention because of the applications to integral operators and partial differential elliptic operators (e.g. in papers of Atiyah-Singer and Atiyah-Bott).

For this second edition, I have added the spectral theorem for unbounded self-adjoint operators. I learned it in connection with the spectral decomposition of the Laplacian on the upper half plane. The bibliography contains references to this literature for those interested.

The fourth part begins with the development of the integral. The fashion has been to emphasize positivity and ordering properties (increasing and decreasing sequences). I find this excessive. The treatment given here attempts to give a proper balance between L^1-convergence and positivity.

The chapters on applications of integration and distributions provide concrete examples and choices for leading the course in other directions, at the taste of the lecturer. There are many very good books in intermediate analysis, and interesting research papers, which can be read immediately after the present course. A partial list is given in the bibliography. In fact, the determination of the material included in this *Real Analysis* has been greatly motivated by the existence of these papers and books, and by the need to provide the necessary background for them.

A number of examples are given in the text (for instance, the Laplace operator in Chapter 8), but many interesting examples are also given in the exercises (for instance, explicit formulas for approximations whose existence one knows abstractly by the Weierstrass-Stone theorem; integral operators of various kinds; etc). The exercises should be viewed as an integral part of the book. Note that Chapter 15, giving the spectral measure, can be viewed as providing an example for many notions which have been discussed previously: operators in Hilbert space, measures, and convolutions. At the same time, these results lead directly into the real analysis of the working mathematician.

For some courses, it will be best to omit a lot of the functional analysis and to cover most of the integration theory. For instance, a course could cover Chapters 2, 3, 4 and §1, §2 of Chapter 7. After that, one could go immediately to integration theory in Chapters 11, 12, and 13. This ordering could make up a single course, even a one semester course if one omits some of the more technical material. Chapter 14 on locally compact spaces would then give a natural continuation. Its purpose is to show how one derives a measure from a functional on the space of continuous functions with compact support. After that, one might cover Chapter 17 on distributions, showing how restrictions on functionals by means of differential operators give rise to a ubiquitous notion in analysis, and also give the flavor of Euclidean space superimposed on the more general measure and functional theory.

I find it appropriate to introduce students to differentiable manifolds during this first year graduate analysis course, not only because these objects are of interest to differential geometers or differential topologists, but because global analysis on manifolds has come into its own, both in its integral and differential aspects. It is therefore desirable to integrate manifolds in analysis courses, and I have done this in the last part, which may also be viewed as providing a good application of integration theory.

As usual, I have avoided as far as possible building long chains of logical interdependence, and have made chapters as logically independent as possible, so that if one wishes to cover integration early, for instance, this can be done without difficulty simply by skipping suitable chapters. My personal taste of the moment was rather to deal with continuous linear algebra first, at some length. I think under any circumstances, a minimum of this "continuous linear algebra" should be done before any other topics (e.g. introducing Banach and Hilbert spaces, and proving the existence of an orthogonal complement in Hilbert space). This gives a language and a mechanism which make everything easier afterward, and is in line with one of the main trends of mathematics, which is to linearize whenever possible.

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Part One

General Topology

CHAPTER 1

Sets

§1. SOME BASIC TERMINOLOGY

We assume that the reader understands the meaning of the word "set", and in this chapter, summarize briefly the basic properties of sets and operations between sets. We denote the empty set by $\emptyset$. A subset S' of S is said to be **proper** if $S' \neq S$. We write $S' \subset S$ or $S \supset S'$ to denote the fact that S' is a subset of S.

Let S, T be sets. A **mapping** $f\colon T \to S$ is an association which to each element $x \in T$ associates an element of S, denoted by $f(x)$, and called the **value** of f at x, or the **image** of x under f. If T' is a subset of T, we denote by $f(T')$ the subset of S consisting of all elements $f(x)$ for $x \in T$. The association of $f(x)$ to x is denoted by the special arrow

$$x \mapsto f(x).$$

Let X, Y be sets. A map $f\colon X \to Y$ is said to be **injective** if for all $x, x' \in X$ with $x \neq x'$ we have $f(x) \neq f(x')$. We say that f is **surjective** if $f(X) = Y$, i.e. if the image of f is all of Y. We say that f is **bijective** if it is both injective and surjective. As usual, one should index a map f by its set of arrival and set of departure to have absolutely correct notation, but this is too clumsy, and the context is supposed to make it clear what these sets are. For instance, let $\mathbf{R}$ denote the real numbers, and $\mathbf{R}'$ the real numbers $\geqq 0$. The map

$$f_{\mathbf{R}}^{\mathbf{R}}\colon \mathbf{R} \to \mathbf{R}$$

given by $x \mapsto x^2$ is not surjective, but the map

$$f_{\mathbf{R}'}^{\mathbf{R}}\colon \mathbf{R} \to \mathbf{R}'$$

given by the same formula is surjective.

If $f: X \to Y$ is a map and S a subset of X, we denote by

$$f | S$$

the restriction of f to S, is the map f viewed as a map defined only on S. For instance, if $f: \mathbf{R} \to \mathbf{R}'$ is the map $x \mapsto x^2$, then f is not injective, but $f|\mathbf{R}'$ is injective.

A composite of injective maps is injective, and a composite of surjective maps is surjective. Hence a composite of bijective maps is bijective.

We denote by $\mathbf{Q}$, $\mathbf{Z}$ the sets of rational numbers and integers respectively. We denote by $\mathbf{Z}^+$ the set of positive integers (integers > 0), and similarly by $\mathbf{R}^+$ the set of positive reals. We denote by $\mathbf{N}$ the set of natural numbers (integers $\geqq 0$), and by $\mathbf{C}$ the complex numbers. A mapping into $\mathbf{R}$ or $\mathbf{C}$ will be called a **function**.

Let S and I be sets. By a **family of elements of** S**, indexed by** I, one means simply a map $f: I \to S$. However, when we speak of a family, we write $f(i)$ as f_i, and also use the notation $\{f_i\}_{i \in I}$ to denote the family.

Example 1. Let S be the set consisting of the single element 3. Let $I = \{1, \ldots, n\}$ be the set of integers from 1 to n. A family of elements of S, indexed by I, can then be written $\{a_i\}_{i=1,\ldots,n}$ with each $a_i = 3$. Note that a family is different from a subset. The same element of S may receive distinct indices.

A family of elements of a set S indexed by positive integers, or nonnegative integers, is also called a **sequence**.

Example 2. A sequence of real numbers is written frequently in the form

$$\{x_1, x_2, \ldots\} \quad \text{or} \quad \{x_n\}_{n \geqq 1}$$

and stands for the map $f: \mathbf{Z}^+ \to \mathbf{R}$ such that $f(i) = x_i$. As before, note that a sequence can have all its elements equal to each other, that is

$$\{1, 1, 1, \ldots\}$$

is a sequence of integers, with $x_i = 1$ for each $i \in \mathbf{Z}^+$.

We define a **family of sets indexed by a set** I in the same manner, that is, a family of sets indexed by I is an assignment

$$i \mapsto S_i$$

which to each $i \in I$ associates a set S_i. The sets S_i may or may not have elements in common, and it is conceivable that they may all be equal. As before, we write the family $\{S_i\}_{i \in I}$.

We can define the intersection and union of families of sets, just as for the intersection and union of a finite number of sets. Thus, if $\{S_i\}_{i \in I}$ is a family of sets, we define the **intersection** of this family to be the set

$$\bigcap_{i \in I} S_i$$

consisting of all elements x which lie in all S_i. We define the **union**

$$\bigcup_{i \in I} S_i$$

to be the set consisting of all x such that x lies in some S_i.

If S, S' are sets, we define $S \times S'$ to be the set of all pairs (x, y) with $x \in S$ and $y \in S'$. We can define finite products in a similar way. If $S_1, S_2, \dots$ is a sequence of sets, we define the product

$$\prod_{i=1}^{\infty} S_i$$

to be the set of all sequences $(x_1, x_2, \dots)$ with $x_i \in S_i$. Similarly, if I is an indexing set, and $\{S_i\}_{i \in I}$ a family of sets, we define the product

$$\prod_{i \in I} S_i$$

to be the set of all families $\{x_i\}_{i \in I}$ with $x_i \in S_i$.

Let X, Y, Z be sets. We have formula

$$(X \cup Y) \times Z = (X \times Z) \cup (Y \times Z).$$

To prove this, let $(w, z) \in (X \cup Y) \times Z$ with $w \in X \cup Y$ and $z \in Z$. Then $w \in X$ or $w \in Y$. Say $w \in X$. Then $(w, z) \in X \times Z$. Thus

$$(X \cup Y) \times Z \subset (X \times Z) \cup (Y \times Z).$$

Conversely, $X \times Z$ is contained in $(X \cup Y) \times Z$ and so is $Y \times Z$. Hence their union is contained in $(X \cup Y) \times Z$, thereby proving our assertion.

We say that two sets X, Y are **disjoint** if their intersection is empty. We say that a union $X \cup Y$ is **disjoint** if X and Y are disjoint. Note that if X, Y are disjoint, then $(X \times Z)$ and $(Y \times Z)$ are disjoint.

We can take products with arbitrary families. For instance, if $\{X_i\}_{i \in I}$ is a family of sets, then

$$\left(\bigcup_{i \in I} X_i\right) \times Z = \bigcup_{i \in I} (X_i \times Z).$$